# Physics and Sources of Gravitational Waves

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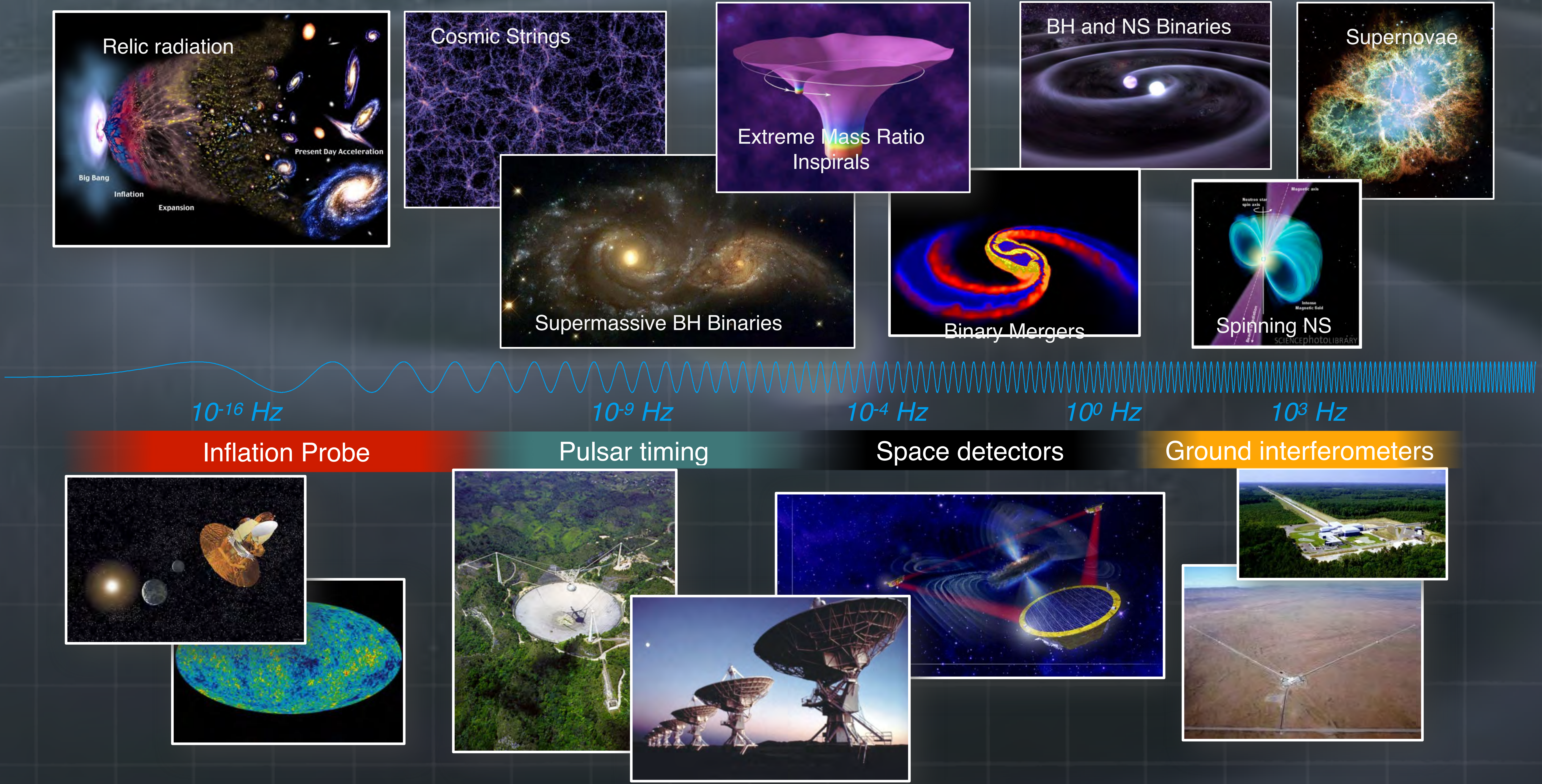
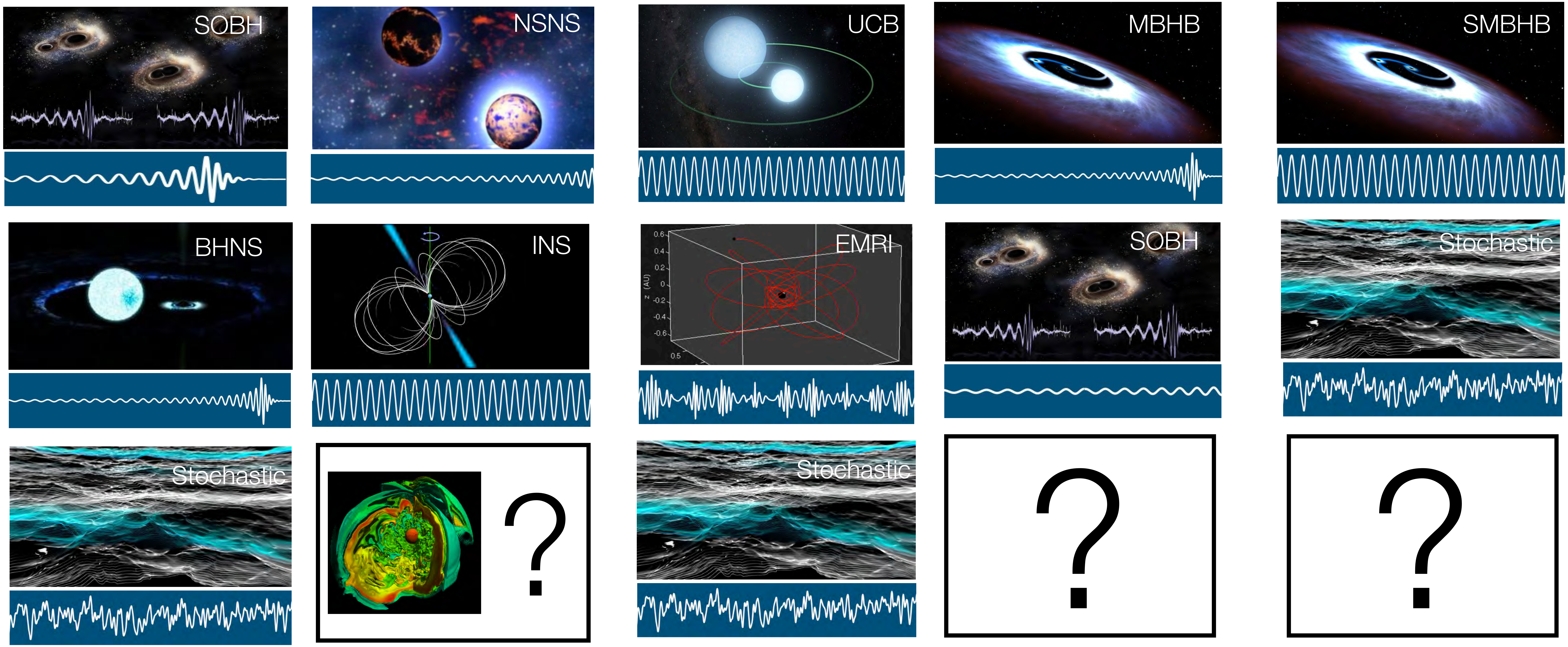
RADIO
MICROWAVE
INFRARED
VISIBLE
ULTRAVIOLET
X-RAY
10 m
10 cm
1 mm
0.3 mm
780 nm
380 nm
10 nm
0.01

# Gravitational Wave Spectrum



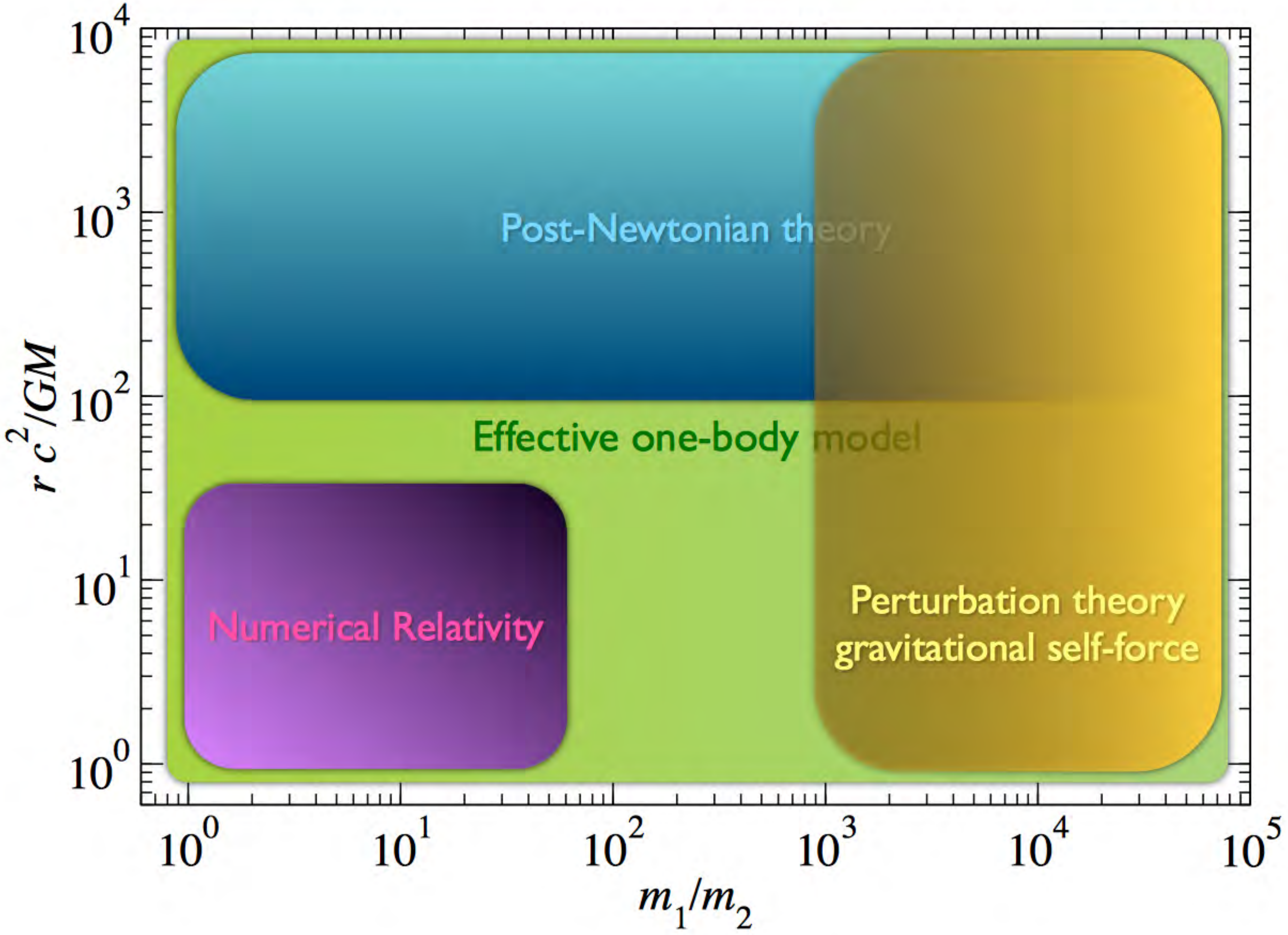
# Gravitational Wave Sources

**LIGO** **LISA** **PTAs**

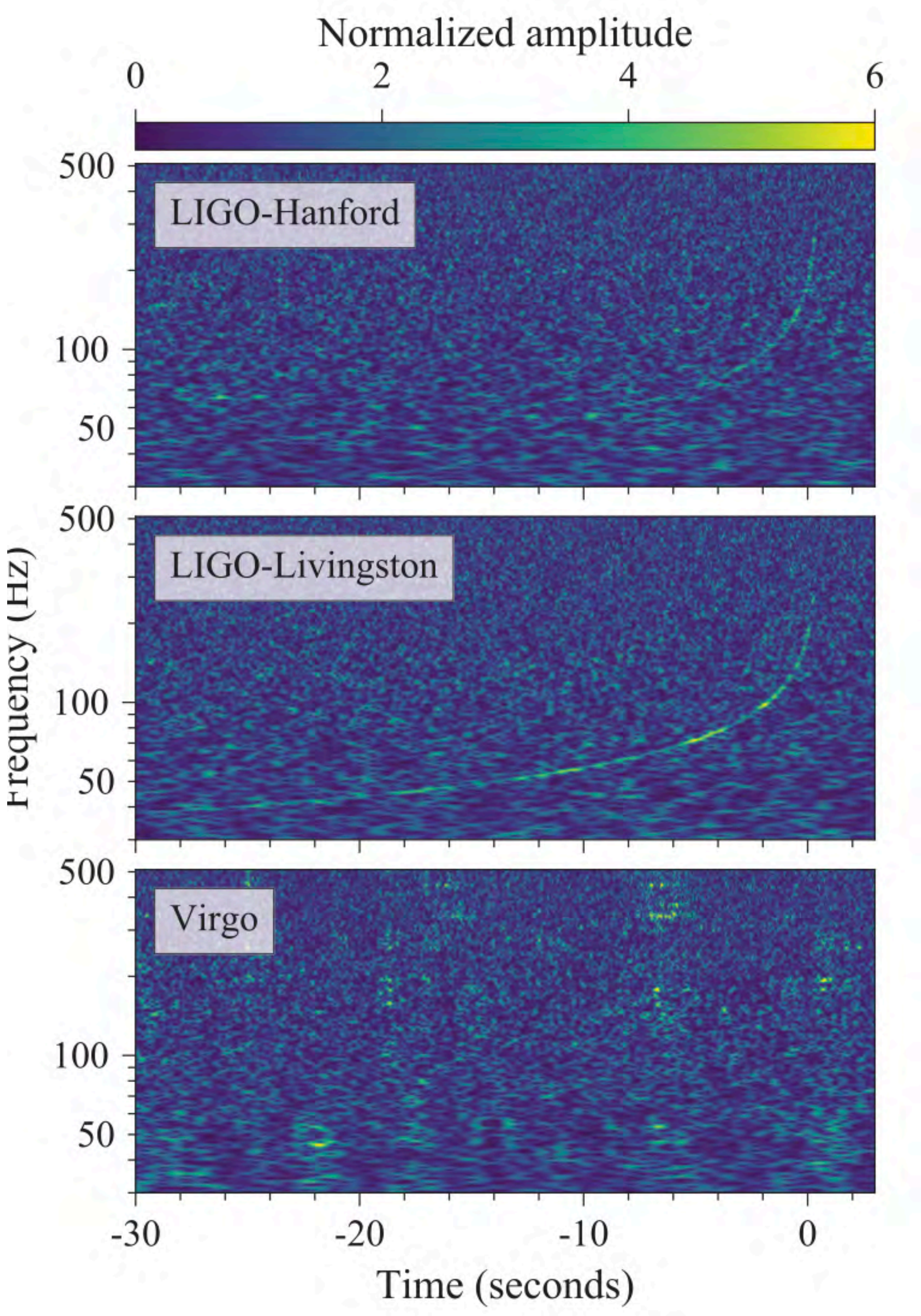


# Solving Einstein's Equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$



# Decoding the Data

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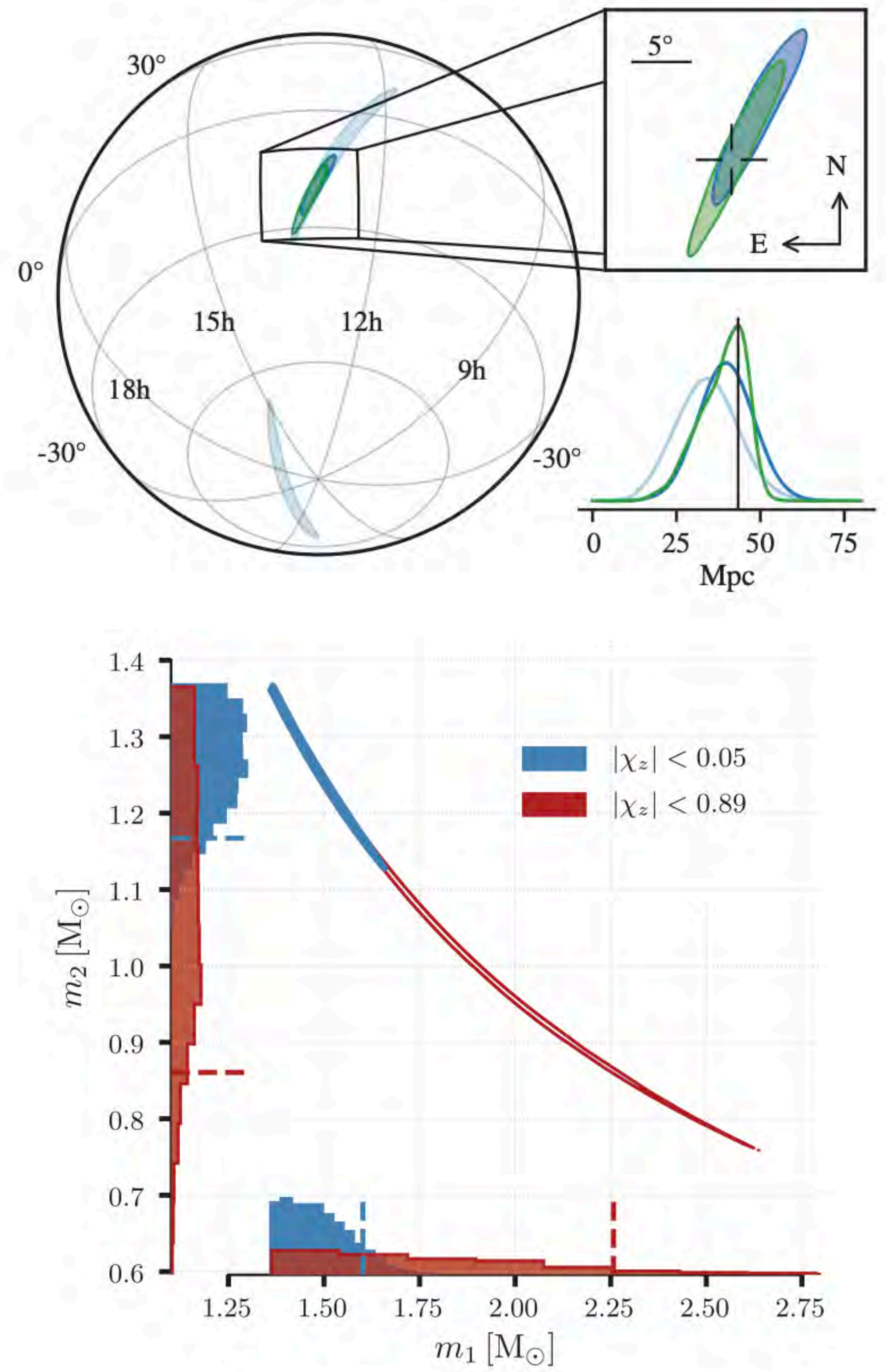
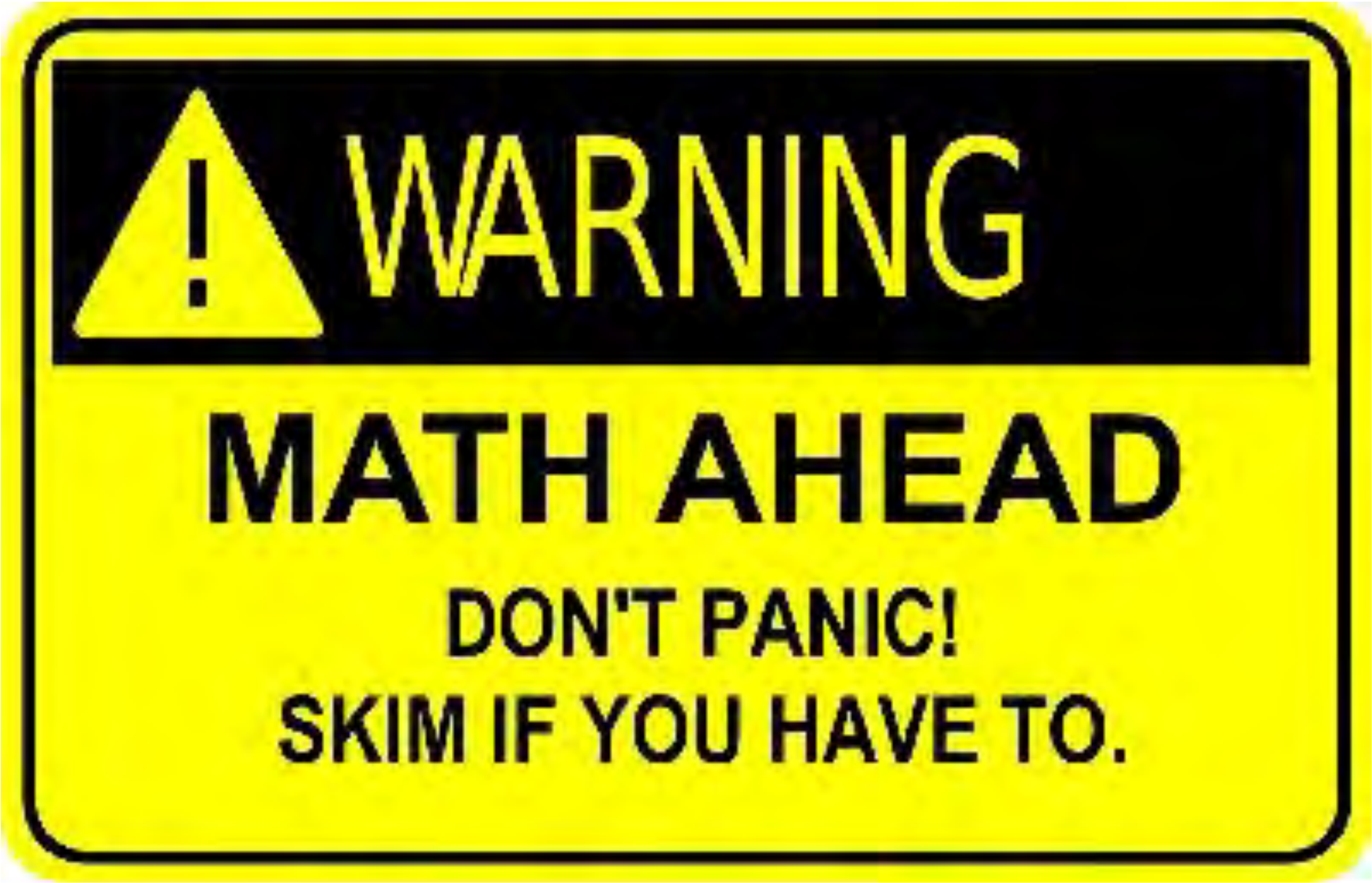


# Part 1. Solving Einstein's Equations



# Solving Einstein's Equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Post-Minkowski Expansion $\quad G \ll 1 \quad (\text{really } U = \frac{GM}{c^2 R} \ll 1)$

Post-Newtonian Expansion $\quad \frac{v}{c} \ll 1$

Black Hole Perturbation/Self Force $\quad q = \frac{m_2}{m_1} \ll 1$

Numerical Relativity $\quad \frac{\partial g}{\partial x} \approx \frac{g(x+\epsilon) - g(x)}{\epsilon}$

# Post-Newtonian Expansion

For gravitationally bound systems $U \sim \frac{v^2}{c^2}$ $\Rightarrow$ PM = PN

Modern treatments of PN theory use the PM formalism since it avoids ambiguities related to retardation

Several different approaches, DIRE, Matched Asymptotic Expansions, EFT

Defining $h^{\alpha\beta} = \eta^{\alpha\beta} - \mathfrak{g}^{\alpha\beta}$ where $\mathfrak{g}^{\alpha\beta} = \sqrt{-g}\, g^{\alpha\beta}$

Einstein's Equations become

$$\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}$$

$$\partial_\beta h^{\alpha\beta} = 0$$

where $\tau^{\alpha\beta} = -g(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL} + t^{\alpha\beta}_{\rm H})$

Both $t^{\alpha\beta}_{\rm LL}, t^{\alpha\beta}_{\rm H}$ are quadratic and higher in $h^{\alpha\beta}$

# Post-Newtonian Expansion

$$\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}$$

$$\partial_\beta h^{\alpha\beta} = 0$$

Formal solution
$$h^{\alpha\beta}(x) = \frac{4G}{c^4}\int \frac{\delta(ct - ct' - |\mathbf{x} - \mathbf{x}'|)\tau^{\alpha\beta}(x')}{|\mathbf{x} - \mathbf{x}'|}\, d^4x'$$

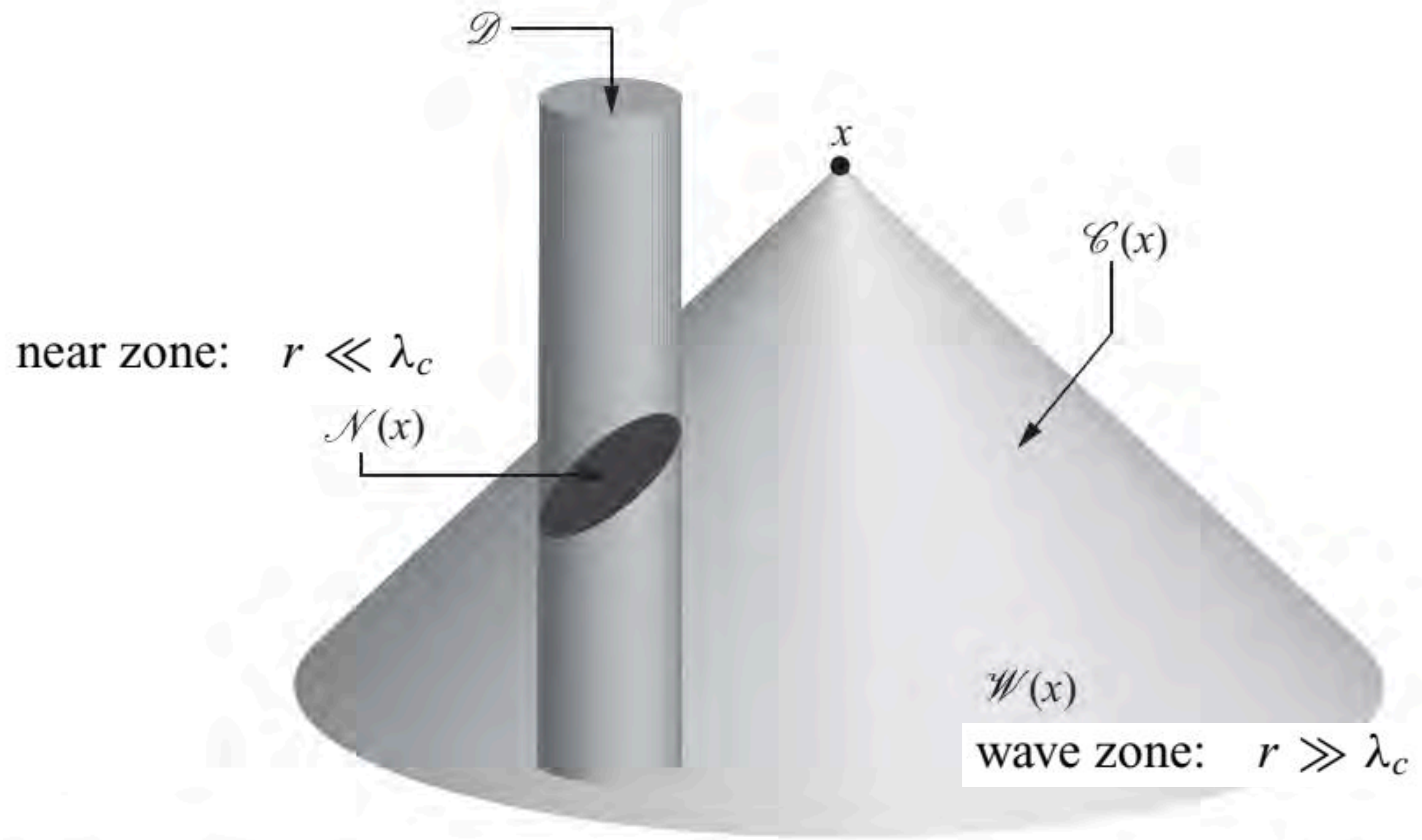
Lends itself to a perturbative solution in powers of $h^n$.

Leading order is just like E&M. Jackson revisited.

Higher orders are trickier because the source is then non-compact.

# Post-Newtonian Expansion

$$h^{\alpha\beta}(x) = \frac{4G}{c^4}\int \frac{\delta(ct - ct' - |\mathbf{x}-\mathbf{x}'|)\tau^{\alpha\beta}(x')}{|\mathbf{x}-\mathbf{x}'|}\, d^4x'$$




Integration domains for the retarded solution of the wave equation: $\mathscr{C}(x)$ is the past light cone of the field point $x$; $\mathscr{D}$ is the world tube traced by a three-dimensional ball of radius $\mathcal{R}$, which contains the near-zone region of spacetime; $\mathscr{N}(x)$ is the intersection of $\mathscr{C}(x)$ with the near zone; and $\mathscr{W}(x)$ is the remaining piece of the light cone.

$t_c :=$ characteristic time scale of the source,

$\omega_c := \frac{2\pi}{t_c} =$ characteristic frequency of the source,

$\lambda_c := \frac{2\pi c}{\omega_c} = ct_c =$ characteristic wavelength of the radiation.

$r_c :=$ characteristic length scale of the compact-support source,

$v_c := \frac{r_c}{t_c} =$ characteristic velocity within the source.

$$v_c \ll c \qquad \Rightarrow \qquad r_c \ll \lambda_c$$

# Post-Newtonian Expansion

$$h^{\alpha\beta}(x) = \frac{4G}{c^4} \int \frac{\delta(ct - ct' - |\mathbf{x} - \mathbf{x}'|)\tau^{\alpha\beta}(x')}{|\mathbf{x} - \mathbf{x}'|} \, d^4x'$$

First iteration yields the solution

$$h^{00} = \frac{4u}{c^2}$$

$$h^{0j} = \frac{4u^j}{c^2}$$

$$h^{jk} = 0$$

$$u(t, \mathbf{x}) = G \int \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d^3x'$$

$$u^j(t, \mathbf{x}) = G \int \frac{\rho(t, \mathbf{x}')v^j(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d^3x'$$

$$\Rightarrow \quad g_{00} = -1 + \frac{2u}{c^2}, \quad g_{0j} = -\frac{4u_j}{c^3}, \quad g_{jk} = \delta_{jk}\left(1 + \frac{2u}{c^2}\right)$$

# Post-Newtonian Expansion

$$h^{\alpha\beta}(x) = \frac{4G}{c^4} \int \frac{\delta(ct - ct' - |\mathbf{x} - \mathbf{x}'|)\tau^{\alpha\beta}(x')}{|\mathbf{x} - \mathbf{x}'|} \, d^4x'$$

Second iteration is sourced by the first. Already the full solution gets a bit messy, with corrections to the metric in the near and wave zone. In the far away wave zone the solution is

$$h^{00} = \frac{4G}{c^2 R}\left(M + \frac{1}{2c^2}\ddot{\mathcal{I}}^{jk} N_j N_k\right) \qquad h^{0j} = \frac{2G}{c^3 R}\ddot{\mathcal{I}}^{jk} N_k \qquad h^{jk} = \frac{2G}{c^4 R}\ddot{\mathcal{I}}^{jk}$$

Mass quadrupole $\quad \mathcal{I}^{jk}(t - r/c) = \int \rho^*(t', x')\, x'^j x'^k \, d^3x' \qquad N^j = \frac{x^j}{R} \qquad R = \sqrt{x^j x_j}$

These expressions are fully general. Can be used for supernova explosions, binary mergers etc

At this stage we have not yet imposed the gauge condition $\partial_\beta h^{\alpha\beta} = 0$

# Post-Newtonian Expansion

Imposing the gauge condition $\partial_\beta h^{\alpha\beta} = 0$ and using the residual gauge freedom to remove the trace of the radiative terms yields

$$h^{00} = \frac{4GM}{c^2 R}$$

$$h^{0j} = 0$$

$$h^{jk} = \frac{2GM}{c^4 R} \Lambda^{jk}_{lm} \ddot{\mathcal{I}}^{lm}$$

Where we have the TT gauge projection tensor $\Lambda^{jk}_{lm} = P^j_l P^k_m - \frac{1}{2} P^{jk} P_{lm}$ with $P^j_k = \delta^j_k - N^j N_k$

The projection tensor ensures that the radiation is transverse and traceless

# Post-Newtonian Expansion

Near zone metric modified - modifies the dynamics. For example, for binary systems

0 PN  1 PN  1.5 PN  2 PN  2 PN  2.5 PN

$$\mathbf{a} = \mathbf{a}_N + \mathbf{a}_{PN} + \mathbf{a}_{SO} + \mathbf{a}_{2PN} + \mathbf{a}_{SS} + \mathbf{a}_{RR} + \ldots$$

Newton

$$\mathbf{a}_N = -\frac{M}{r^2}\,\hat{\mathbf{n}}$$

Perihelion precession

$$\mathbf{a}_{PN} = -\frac{M}{r^2}\left\{\hat{\mathbf{n}}\left[(1+3\eta)v^2 - 2(2+\eta)\frac{M}{r} - \frac{3}{2}\eta\dot{r}^2\right] - 2(2-\eta)\dot{r}\mathbf{v}\right\}$$

Spin-orbit coupling

$$\mathbf{a}_{SO} = \frac{1}{r^3}\left\{6\hat{\mathbf{n}}[(\hat{\mathbf{n}}\times\mathbf{v})\cdot(2\mathbf{S} + \frac{\delta M}{M}\boldsymbol{\Delta})] - [\mathbf{v}\times(7\mathbf{S} + 3\frac{\delta M}{M}\boldsymbol{\Delta})] + 3\dot{r}[\hat{\mathbf{n}}\times(3\mathbf{S} + \frac{\delta M}{M}\boldsymbol{\Delta})]\right\}$$

Gravity gravitates

$$\mathbf{a}_{2PN} = -\frac{M}{r^2}\left\{\hat{\mathbf{n}}\left[\frac{3}{4}(12+29\eta)\left(\frac{M}{r}\right)^2 + \eta(3-4\eta)v^4 + \frac{15}{8}\eta(1-3\eta)\dot{r}^4 \right.\right.$$
$$\left. -\frac{3}{2}\eta(3-4\eta)v^2\dot{r}^2 - \frac{1}{2}\eta(13-4\eta)\frac{M}{r}v^2 - (2+25\eta+2\eta^2)\frac{M}{r}\dot{r}^2\right]$$
$$\left. -\frac{1}{2}\dot{r}\mathbf{v}\left[\eta(15+4\eta)v^2 - (4+41\eta+8\eta^2)\frac{M}{r} - 3\eta(3+2\eta)\dot{r}^2\right]\right\}$$

# Post-Newtonian Expansion

| 0 PN | 1 PN | 1.5 PN | 2 PN | 2 PN | 2.5 PN |
|---|---|---|---|---|---|

$$\mathbf{a} = \mathbf{a}_N + \mathbf{a}_{PN} + \mathbf{a}_{SO} + \mathbf{a}_{2PN} + \mathbf{a}_{SS} + \mathbf{a}_{RR} + \ldots$$

Spin-spin coupling

$$\mathbf{a}_{SS} = -\frac{3}{\mu r^4}\left\{\hat{\mathbf{n}}(\mathbf{S_1}\cdot\mathbf{S_2}) + \mathbf{S_1}(\hat{\mathbf{n}}\cdot\mathbf{S_2}) + \mathbf{S_2}(\hat{\mathbf{n}}\cdot\mathbf{S_1}) - 5\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{S_1})(\hat{\mathbf{n}}\cdot\mathbf{S_2})\right.$$

Orbital decay

$$\mathbf{a}_{RR} = \frac{8}{5}\eta\frac{M^2}{r^3}\left\{\dot{r}\hat{\mathbf{n}}\left[18v^2 + \frac{2}{3}\frac{M}{r} - 25\dot{r}^2\right] - \mathbf{v}\left[6v^2 - 2\frac{M}{r} - 15\dot{r}^2\right]\right\}$$

Now heroically continued to 4 PN order by Luc Blanchet's group: [Marchand, Bernard, Blanchet & Faye, arXiv:1707.09289 (2017)]

Note on notation:

$$M = m_1 + m_2 \qquad \mu = \frac{m_1 m_2}{M} \qquad \eta = \frac{\mu}{M} = \frac{m_1 m_2}{M^2}$$

$$\delta M = m_1 - m_2 \qquad \mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 \qquad \mathbf{\Delta} = M\left(\frac{\mathbf{S_2}}{m_2} - \frac{\mathbf{S_1}}{m_1}\right)$$

# Post-Newtonian Expansion

We also have equations for the spin and orbital angular momentum evolution at 2PN:

$$\dot{\mathbf{S}}_{\mathbf{1}} = \frac{1}{r^3}\left\{(\mathbf{L_N} \times \mathbf{S_1})(2 + \frac{3}{2}\frac{m_2}{m_1}) - \mathbf{S_2} \times \mathbf{S_1} + 3(\hat{\mathbf{n}} \cdot \mathbf{S_2})\hat{\mathbf{n}} \times \mathbf{S_1}\right\}$$

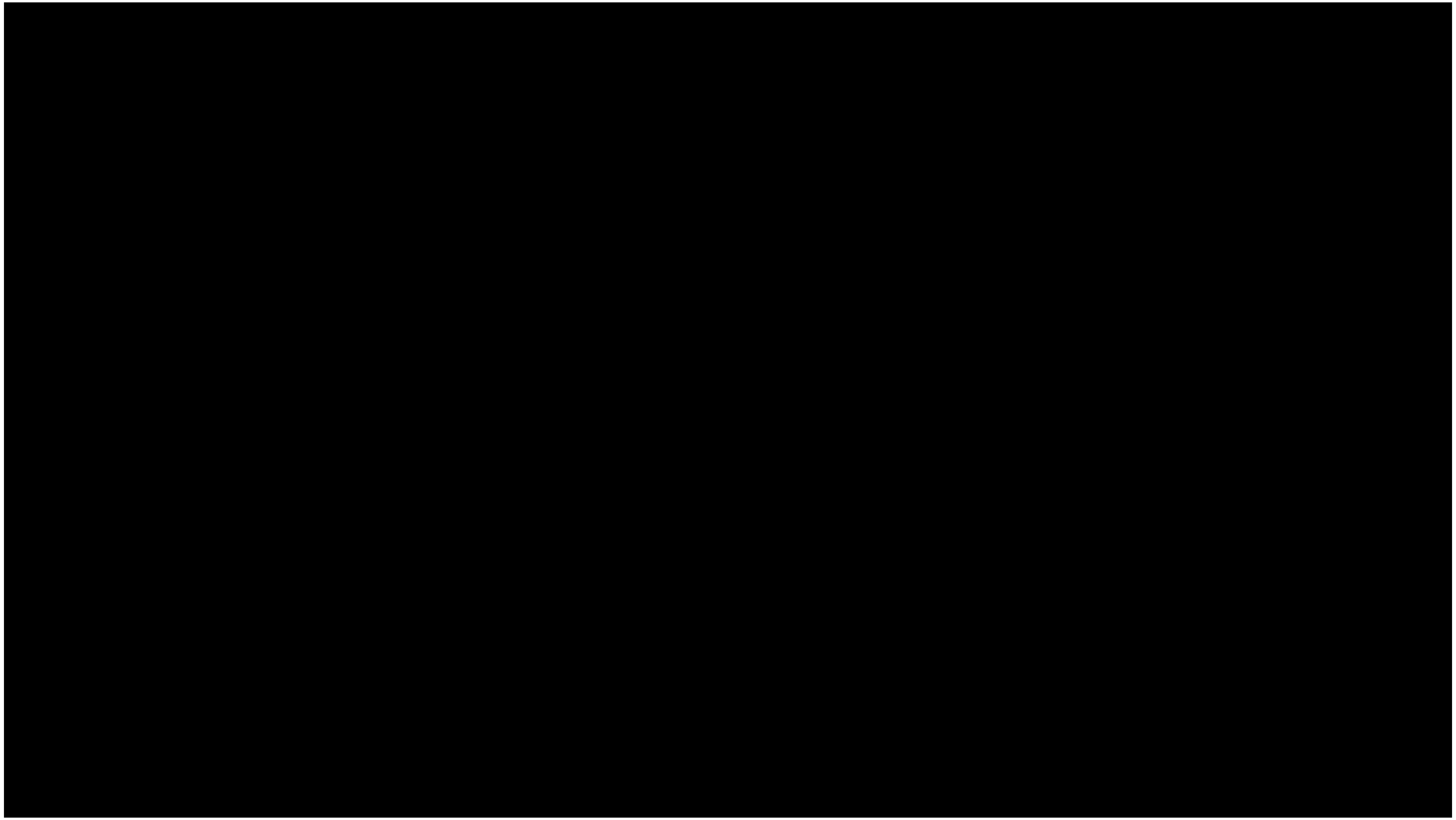
$$\dot{\mathbf{S}}_{\mathbf{2}} = \frac{1}{r^3}\left\{(\mathbf{L_N} \times \mathbf{S_2})(2 + \frac{3}{2}\frac{m_1}{m_2}) - \mathbf{S_1} \times \mathbf{S_2} + 3(\hat{\mathbf{n}} \cdot \mathbf{S_1})\hat{\mathbf{n}} \times \mathbf{S_2}\right\}$$

$$\dot{\mathbf{L}} = -\frac{1}{r^3}\left\{\left[\mathbf{L_N} \times \left(\frac{7}{2}\mathbf{S} + \frac{3}{2}\frac{\delta M}{M}\mathbf{\Delta}\right)\right] + 3(\hat{\mathbf{n}} \cdot \mathbf{S_1})(\hat{\mathbf{n}} \times \mathbf{S_2}) + 3(\hat{\mathbf{n}} \cdot \mathbf{S_2})(\hat{\mathbf{n}} \times \mathbf{S_1})\right\}$$

At 2 PN order the dynamics is conservative: $\mathbf{J} = \mathbf{L} + \mathbf{S} = \text{constant}$

Exact solution for circular binary: [Kesden, Gerosa, O'Shaughnessy, Berti, Sperhake, Phys. Rev. Lett. 114, 081103 (2015)]
[Chatziioannou, Klein, Yunes, Cornish, Phys. Rev. Lett. 118, 051101 (2017)]

# Spin precession

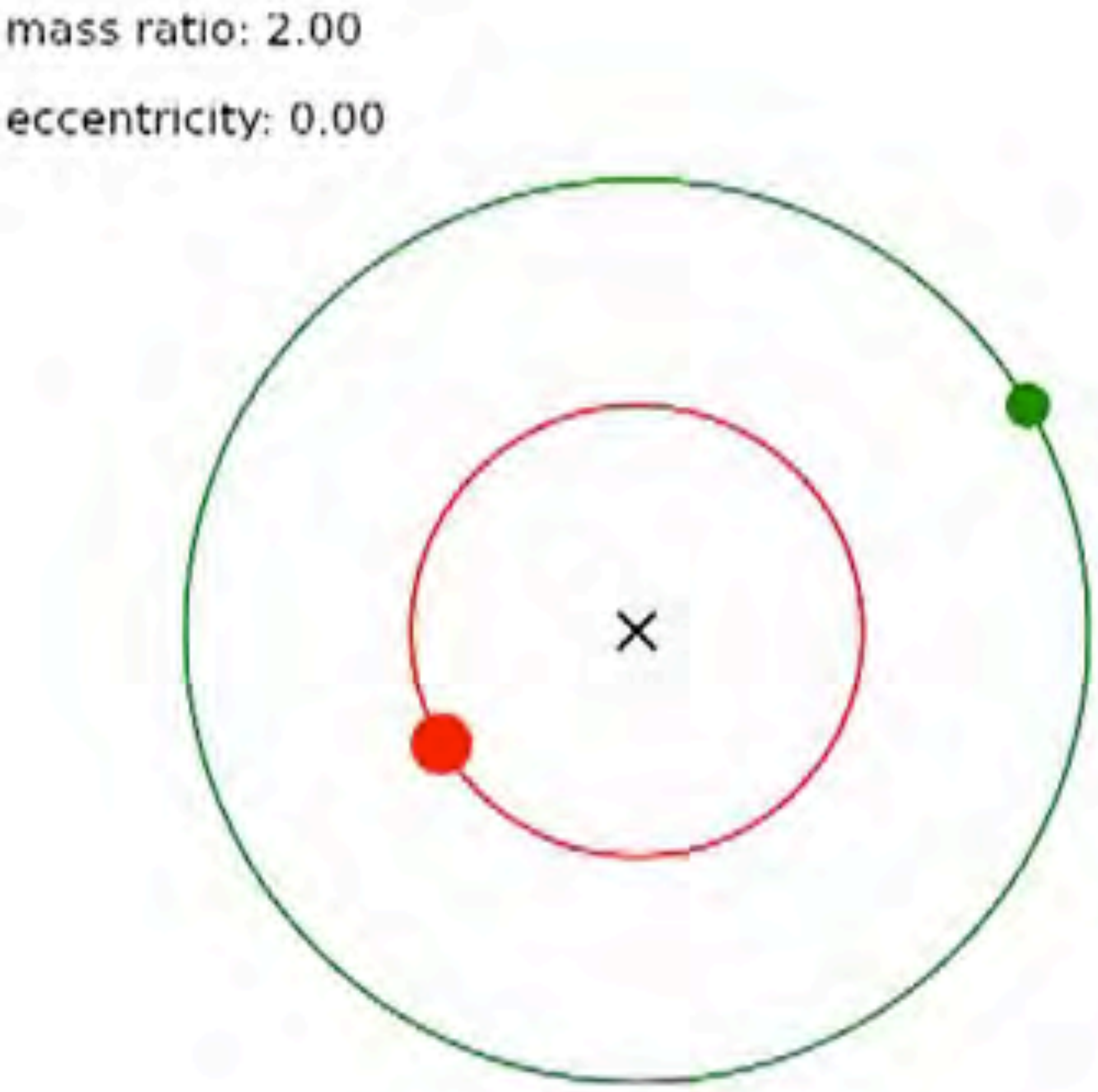


$$\dot{\mathbf{S}}_{\mathbf{1}} = \frac{1}{r^3}\left\{(\mathbf{L_N}\times\mathbf{S_1})(2+\frac{3}{2}\frac{m_2}{m_1}) - \mathbf{S_2}\times\mathbf{S_1} + 3(\hat{\mathbf{n}}\cdot\mathbf{S_2})\hat{\mathbf{n}}\times\mathbf{S_1}\right\}$$

$$\dot{\mathbf{S}}_{\mathbf{2}} = \frac{1}{r^3}\left\{(\mathbf{L_N}\times\mathbf{S_2})(2+\frac{3}{2}\frac{m_1}{m_2}) - \mathbf{S_1}\times\mathbf{S_2} + 3(\hat{\mathbf{n}}\cdot\mathbf{S_1})\hat{\mathbf{n}}\times\mathbf{S_2}\right\}$$

$$\dot{\mathbf{L}} = -\frac{1}{r^3}\left\{\left[\mathbf{L_N}\times\left(\frac{7}{2}\mathbf{S}+\frac{3}{2}\frac{\delta M}{M}\boldsymbol{\Delta}\right)\right] + 3(\hat{\mathbf{n}}\cdot\mathbf{S_1})(\hat{\mathbf{n}}\times\mathbf{S_2}) + 3(\hat{\mathbf{n}}\cdot\mathbf{S_2})(\hat{\mathbf{n}}\times\mathbf{S_1})\right\}$$

# Circular Newtonian Binary




$$\mathbf{x}_{\rm COM} = \frac{m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2}{M} \qquad \mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2 = r\,\hat{\mathbf{n}}$$

$$\frac{d^2\mathbf{x}}{dt^2} = -\frac{M}{r^2}\,\hat{\mathbf{n}}$$

Circular orbit in x-y plane

$$\mathbf{x} = -r\sin(\omega t)\,\hat{i} + r\cos(\omega t)\,\hat{j}$$

$$\omega^2 r^3 = M$$

# Circular Newtonian Binary

Location of masses in COM frame:

$$\mathbf{x}_1 = \frac{m_2}{M}\left(-r\sin(\omega t)\,\hat{i} + r\cos(\omega t)\,\hat{j}\right) \qquad \mathbf{x}_2 = \frac{m_1}{M}\left(r\sin(\omega t)\,\hat{i} - r\cos(\omega t)\,\hat{j}\right)$$

$$\mathcal{I}^{ij} = \int \rho(t,\mathbf{x}) x^i x^j \, d^3x$$

Mass quadrupole

$$\rho(t,\mathbf{x}) = m_1\delta(\mathbf{x}-\mathbf{x}_1) + m_2\delta(\mathbf{x}-\mathbf{x}_2)$$

Mass density

$$\mathcal{I}^{xx} = \mu r^2 \sin^2(\omega t) \qquad \mathcal{I}^{zz} = 0$$

$$\mathcal{I}^{yy} = \mu r^2 \cos^2(\omega t) \qquad \mathcal{I}^{xy} = -\mu r^2 \sin(2\omega t)$$

# Circular Newtonian Binary

$$\ddot{\mathcal{I}}^{xx} = 2\mu r^2\omega^2\cos(2\omega t) = \frac{2\mu M}{r}\cos(2\omega t) \qquad \ddot{\mathcal{I}}^{xy} = 4\mu r^2\omega^2\sin(2\omega t) = \frac{4\mu M}{r}\sin(2\omega t)$$

$$\ddot{\mathcal{I}}^{yy} = -2\mu r^2\omega^2\cos(2\omega t) = -\frac{2\mu M}{r}\cos(2\omega t) \qquad \ddot{\mathcal{I}}^{zz} = 0$$

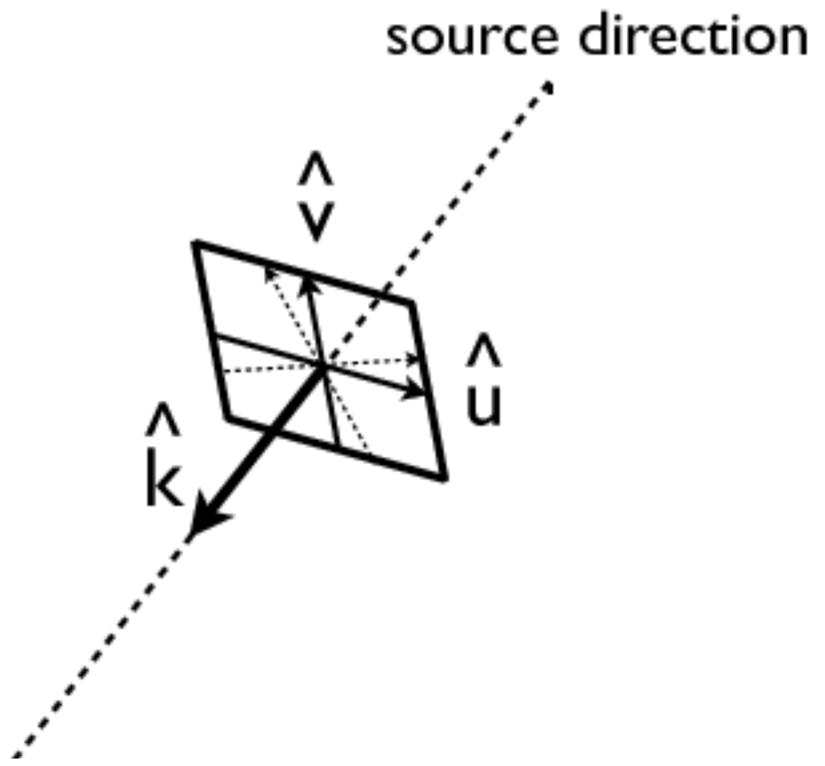
$$h_{ij}^{TT}(t,\mathbf{x}) = \frac{2}{r}\Lambda_{ijkl}\ddot{\mathcal{I}}^{kl}(t-r) \qquad \Lambda_{ijkl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \qquad P_{ij} = \delta_{ij} - k_i k_j$$

Propagation direction $\mathbf{k} = -(\cos\phi\sin\theta\,\hat{i} + \sin\phi\sin\theta\,\hat{j} + \cos\theta\,\hat{k})$

Applying the projection tensor leaves **h** with components in the plane perpendicular to **k** spanned by **u** and **v**:

$$\mathbf{u} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{z}$$

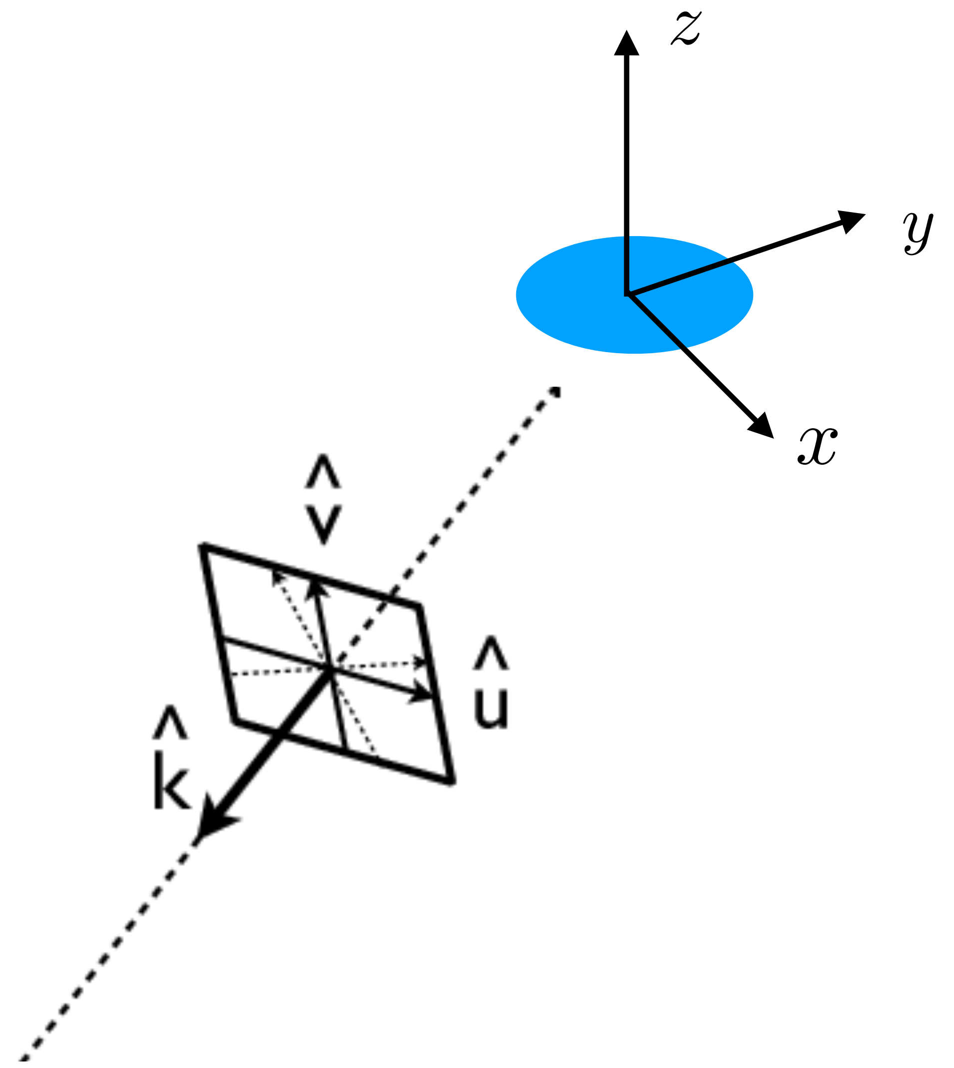
$$\mathbf{v} = \sin\phi\,\hat{i} - \cos\phi\,\hat{j}$$



# Quasi-Circular Binary, Leading Order

$$h_{uu} = -h_{vv} = h_+ = \frac{2\mu M}{r\,R}\left(1+\cos^2\theta\right)\cos(2\omega t + 2\phi)$$

$$h_{uv} = h_\times = \frac{4\mu M}{r\,R}\cos\theta\sin(2\omega t + 2\phi)$$



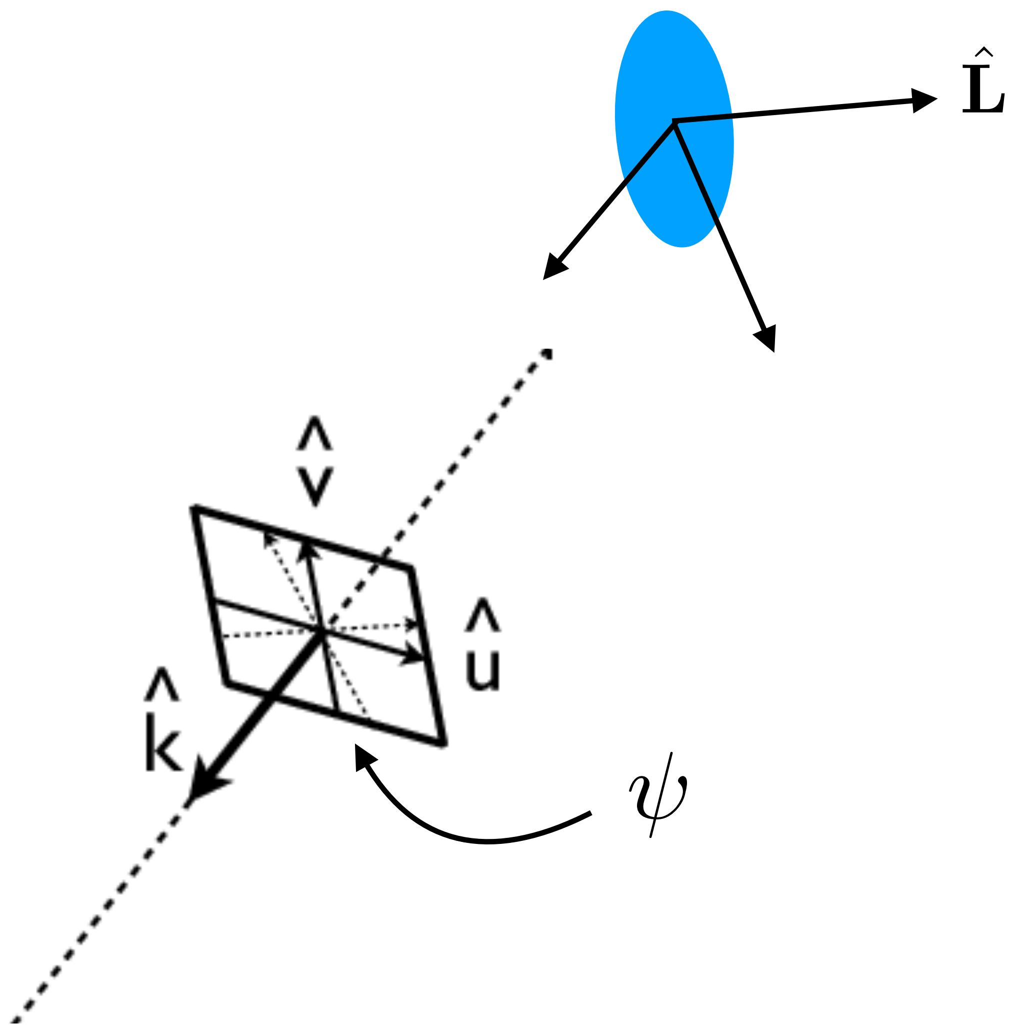

For a binary with a general orientation of the orbital plane

$$h_{uu} = -h_{vv} = \cos(2\psi)h_+ + \sin(2\psi)h_\times$$

$$h_{uv} = -\sin(2\psi)h_+ + \cos(2\psi)h_\times$$

$$h_+ = \frac{2\mu M}{rR}(1+\cos^2\iota)\cos(2\omega t) \qquad h_\times = \frac{4\mu M}{rR}\cos\iota\,\sin(2\omega t)$$

$$\cos\iota = -\hat{k}\cdot\hat{\mathbf{L}} \qquad \tan\psi = \frac{\hat{\mathbf{L}}\cdot\hat{\mathbf{z}} - (\hat{\mathbf{L}}\cdot\hat{\mathbf{k}})(\hat{\mathbf{z}}\cdot\hat{\mathbf{k}})}{\hat{\mathbf{k}}\cdot(\hat{\mathbf{L}}\times\hat{\mathbf{z}})}$$



# Quasi-Circular Binary, Leading Order

$$h_+ = \frac{2\mu M}{rR}(1+\cos^2\iota)\cos(2\omega t) \qquad h_\times = \frac{4\mu M}{rR}\cos\iota\,\sin(2\omega t)$$

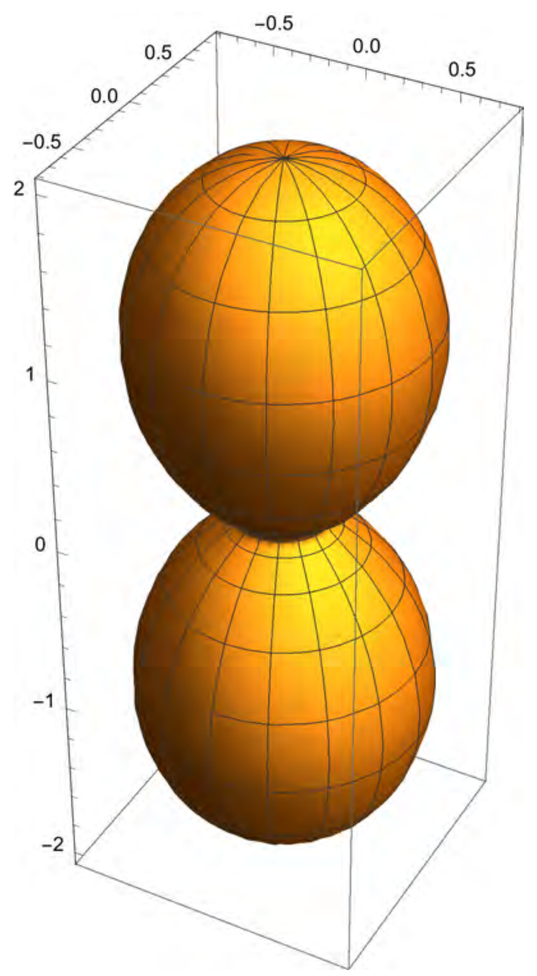
Using $\omega^2 r^3 = M$ we can eliminate the orbital radius *r*

$$h_+ = \frac{2}{R}\mathcal{M}^{5/3}\omega^{2/3}(1+\cos^2\iota)\cos(2\omega t) \qquad h_\times = \frac{4}{R}\mathcal{M}^{5/3}\omega^{2/3}\cos\iota\,\sin(2\omega t)$$

Chirp mass: $\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{M^{1/5}}$

# Energy Emission: Quasi-Circular Binary

$$\left(\frac{dP}{d\Omega}\right)_{\rm quad} = \frac{R^2}{16\pi}\langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle = \frac{2\mu^2 r^4 \omega^6}{\pi}\left(\left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta\right)$$



$$\Rightarrow \quad P_{\rm quad} = \frac{32\mu^2 r^4 \omega^6}{5} = \frac{32}{5}\left(\mathcal{M}\omega\right)^{10/3} = \frac{32}{5}\eta^2 v^{10}$$

(Note: 2.5PN order effect)

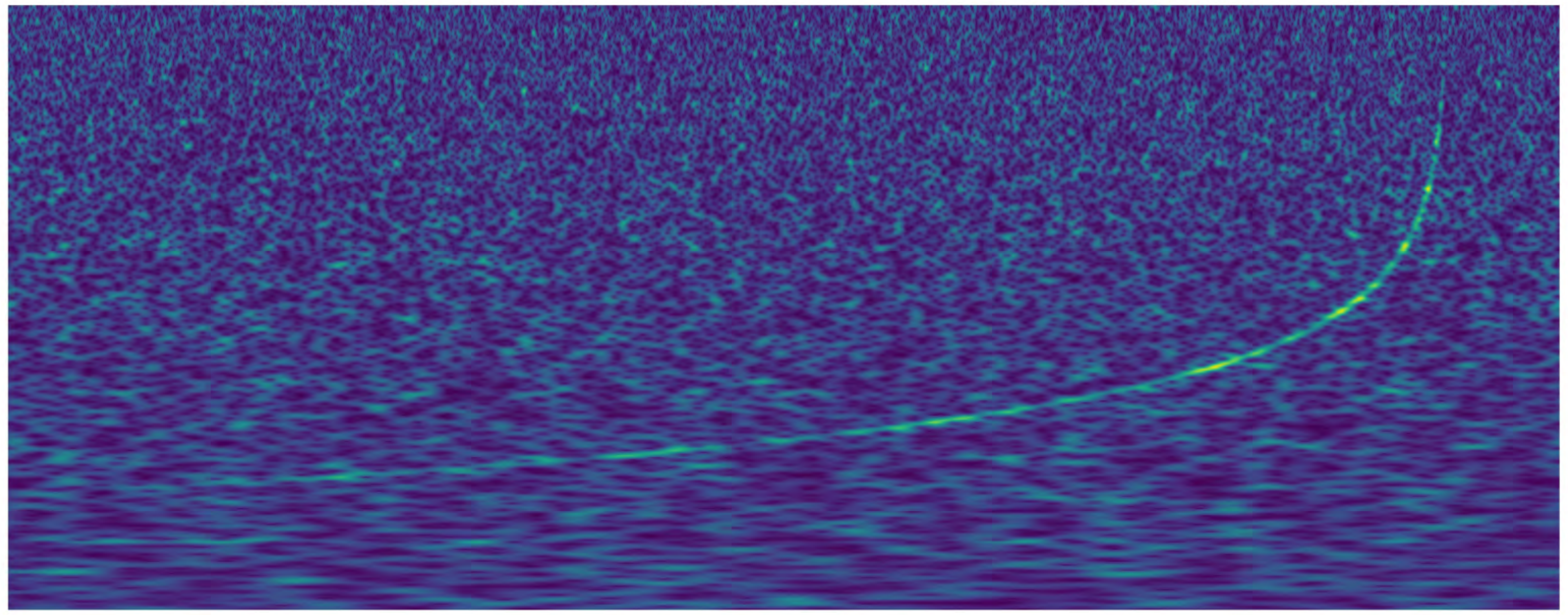
Orbital Energy: $$E = \frac{1}{2}\mu v^2 - \frac{\mu M}{r} = -\frac{\mu M}{2r} = -\frac{1}{2}\mathcal{M}^{5/3}\omega^{2/3}$$

Orbital Decay: $$\frac{dE}{dt} = -P_{\rm quad} \qquad \Rightarrow \qquad \dot{\omega} = \frac{96}{5}\mathcal{M}^{5/3}\omega^{11/3}$$

# Quasi-Circular Binary, Phase Evolution

$$\dot{\omega} = \frac{96}{5}\mathcal{M}^{5/3}\omega^{11/3} \qquad \Rightarrow \quad \int \frac{d\omega}{\omega^{\frac{11}{3}}} = \frac{96}{5}\mathcal{M}^{5/3}\int dt$$

$$\Rightarrow \quad \omega(t) = \frac{1}{\mathcal{M}}\left(\frac{5\mathcal{M}}{256(t_c - t)}\right)^{3/8}$$



$$\Phi(t) = \int 2\omega(t)\, dt = \Phi_c - \frac{1}{16}\left(\frac{256(t_c - t)}{5\mathcal{M}}\right)^{5/8}$$

# Quasi-Circular Binary, Phase Evolution

$$\dot{\omega} = \frac{96}{5}\mathcal{M}^{5/3}\omega^{11/3} \qquad \Rightarrow \quad \int \frac{d\omega}{\omega^{\frac{11}{3}}} = \frac{96}{5}\mathcal{M}^{5/3}\int dt$$

$$\Rightarrow \quad \omega(t) = \frac{1}{\mathcal{M}}\left(\frac{5\mathcal{M}}{256(t_c - t)}\right)^{3/8}$$

$$\Phi(t) = \int 2\omega(t)\,dt = \Phi_c - \frac{1}{16}\left(\frac{256(t_c - t)}{5\mathcal{M}}\right)^{5/8}$$

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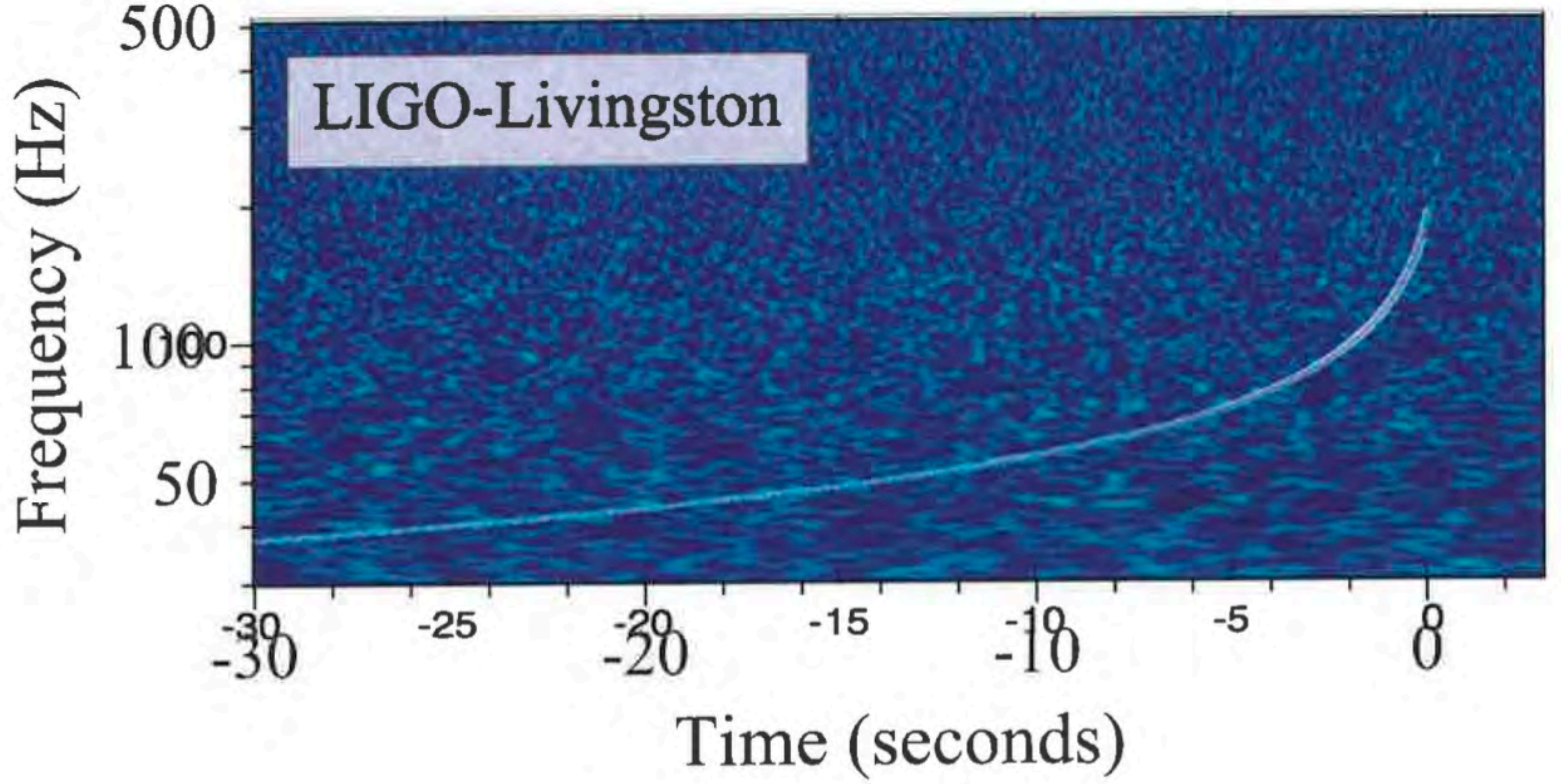
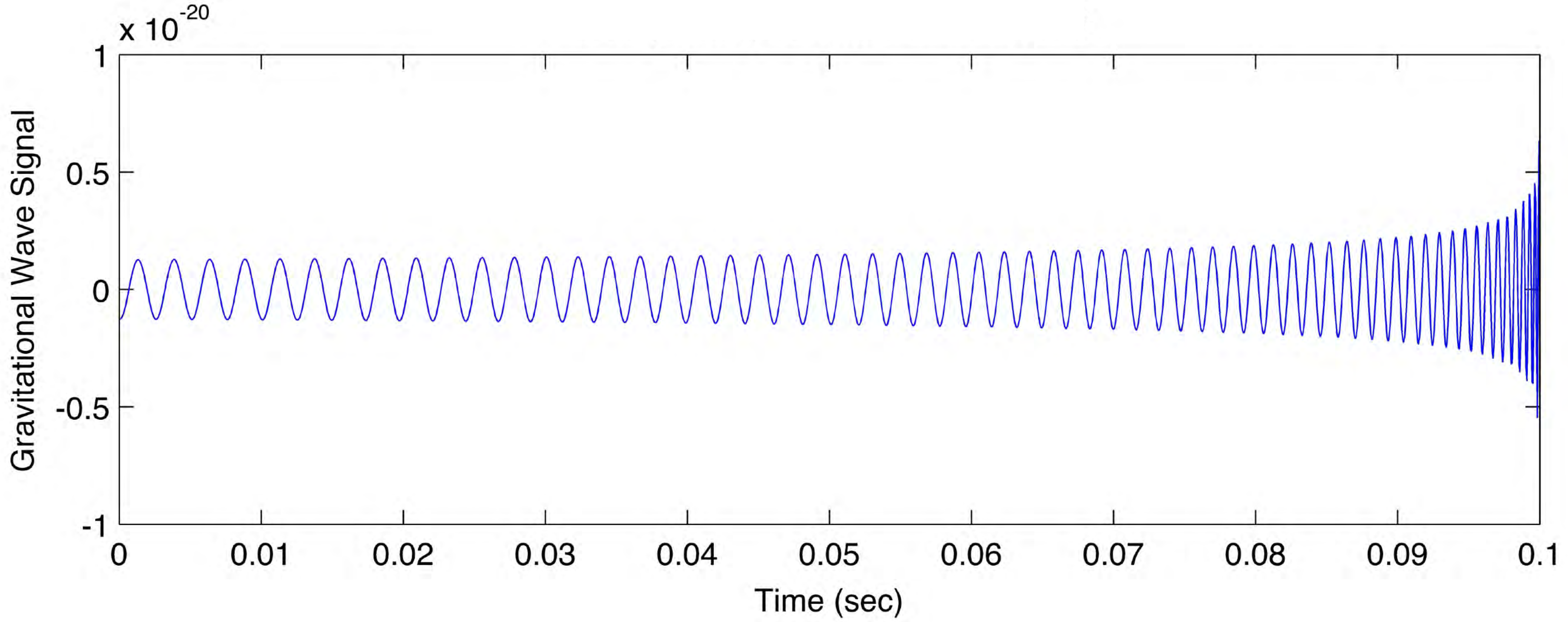



$$\mathcal{M} = 1.19 \pm 0.04\ M_{\odot}$$

$$t_c = 1187008882.4 \pm 0.1\ s$$

# Gravitational Wave Chirp

$$h(t) = \frac{4\mathcal{M}}{R} \left(\mathcal{M}\omega(t)\right)^{2/3} e^{-i\Phi(t)}$$




$$\omega(t) = \frac{1}{\mathcal{M}} \left(\frac{5\mathcal{M}}{256(t_c - t)}\right)^{3/8}$$

$$\Phi(t) = \Phi_c - \frac{1}{16} \left(\frac{256(t_c - t)}{5\mathcal{M}}\right)^{5/8}$$

# Geodesic Deviation

$$\frac{d^2\zeta^\alpha}{d\tau^2} = R^\alpha_{\ \mu\nu\beta}U^\mu U^\nu \zeta^\beta$$

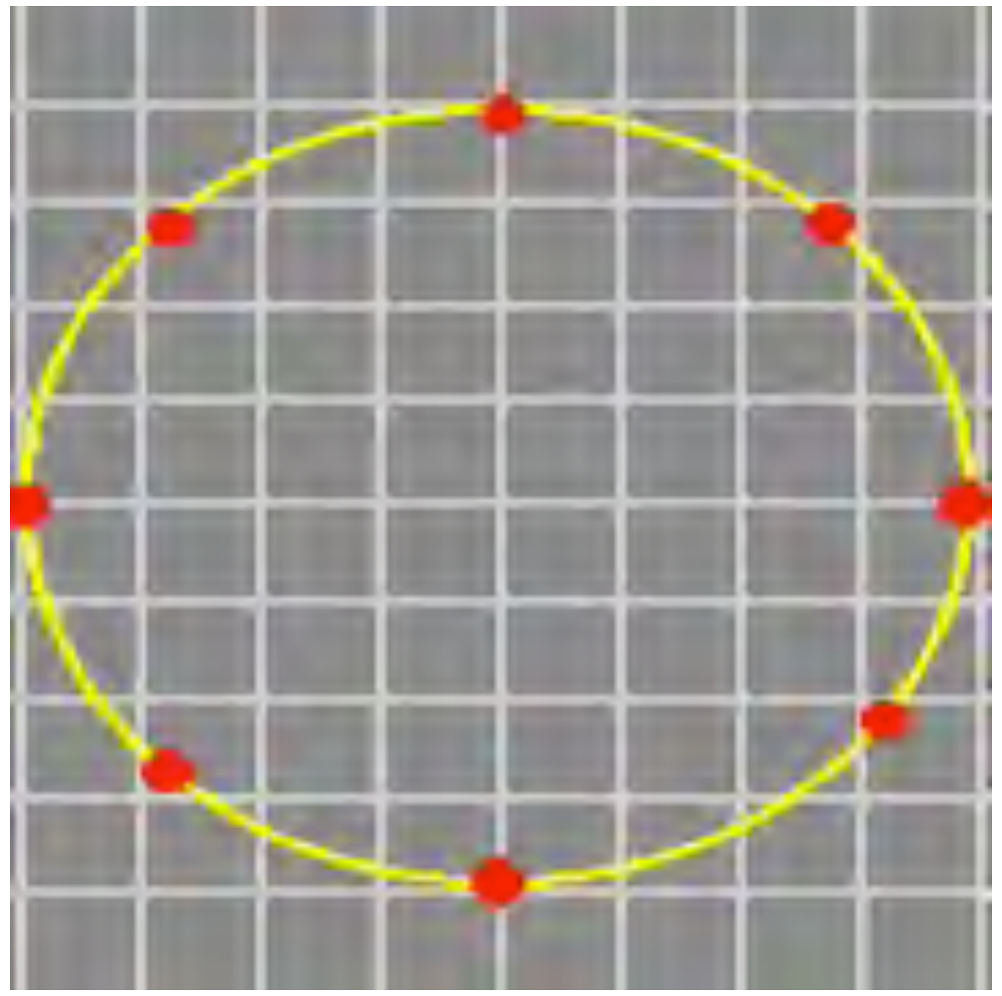
Ring of test particles in the x-y plane, GW propagating in the z direction

$$\frac{d^2\zeta^i}{dt^2} = R^i_{\ 00j}\zeta^j$$

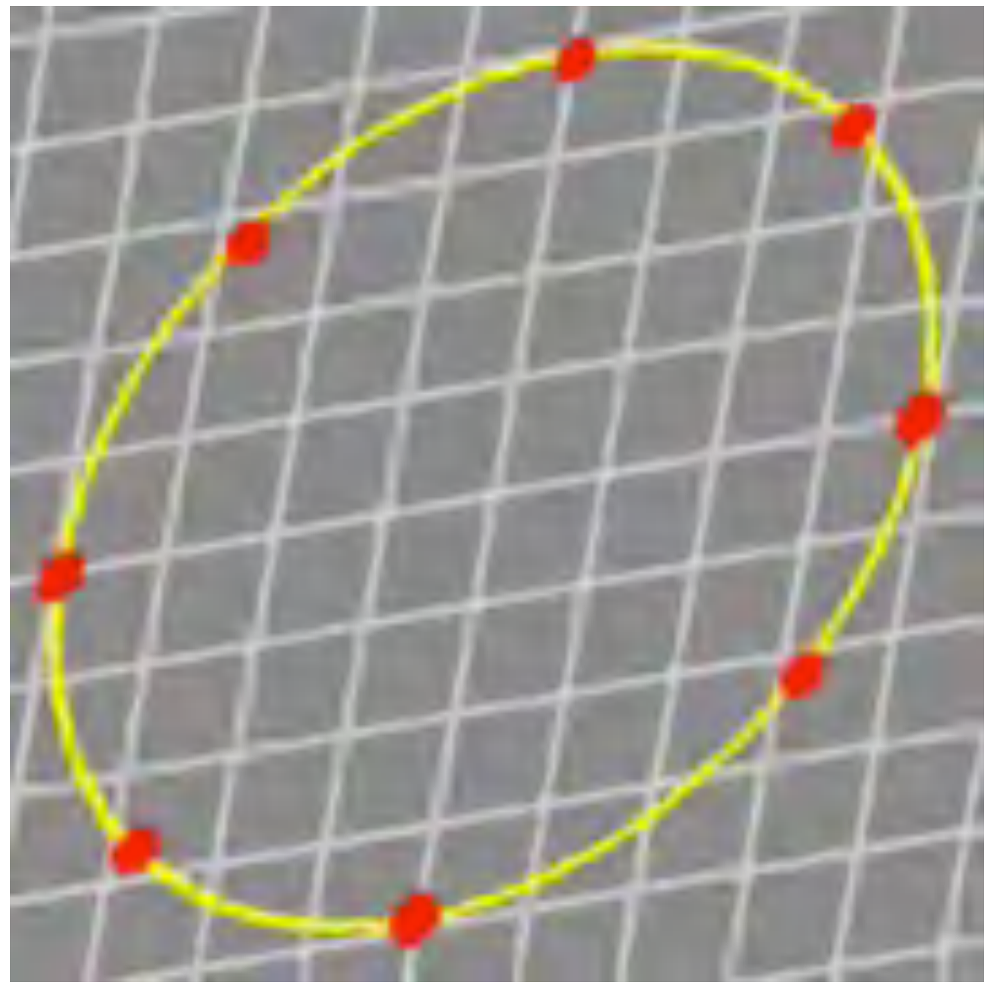
$$R^x_{\ 00x} = -R^y_{\ 00y} = -\frac{\omega^2}{2}h_+$$

$$R^x_{\ 00y} = R^y_{\ 00x} = -\frac{\omega^2}{2}h_\times$$

$h_+$



$h_\times$



# Stationary Phase Approximation

$$h(t) = \frac{4\mathcal{M}}{R}\left(\mathcal{M}\omega(t)\right)^{2/3} e^{-i\Phi(t)}$$

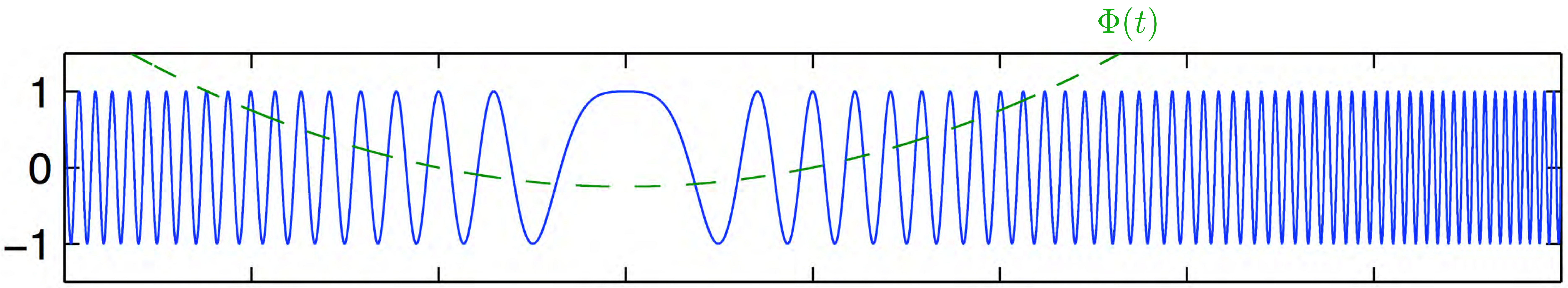
$$\omega(t) = \frac{1}{\mathcal{M}}\left(\frac{5\mathcal{M}}{256(t_c - t)}\right)^{3/8}$$

$$\Phi(t) = \Phi_c - \frac{1}{16}\left(\frac{256(t_c - t)}{5\mathcal{M}}\right)^{5/8}$$

LIGO analyses conducted in frequency domain, need to Fourier transform

$$\tilde{h}(f) = \int dt\, h(t)\, e^{i2\pi f t} = \frac{4\mathcal{M}^{5/3}}{R}\int dt\, \omega(t)^{2/3} e^{i(2\pi f t - \Phi(t))}$$

Rapidly oscillating and averaging to zero unless $\dot{\Phi} = 2\pi f$



# Stationary Phase Approximation

$$\tilde{h}(f) = \int dt\, h(t)\, e^{i2\pi f t} = \frac{4\mathcal{M}^{5/3}}{R} \int dt\, \omega(t)^{2/3} e^{i(2\pi f t - \Phi(t))}$$

Taylor expand around stationary point: $\Phi(t) = \Phi(t_*) + \dot{\Phi}(t_*)(t - t_*) + \frac{1}{2}\ddot{\Phi}(t_*)(t - t_*)^2 + \ldots$ $\qquad \dot{\Phi}(t_*) = 2\pi f$

$$\tilde{h}(f) = \frac{4\mathcal{M}^{5/3}}{R} e^{i(2\pi f t_* - \Phi(t_*))} \omega(t_*)^{2/3} \int dt\, e^{-\frac{i}{2}\ddot{\Phi}(t_*)(t-t_*)^2}$$

$$= \frac{4\mathcal{M}^{5/3}}{R} e^{i(2\pi f t_* - \Phi(t_*) - \pi/4)} \omega(t_*)^{2/3} \left( \frac{2\pi}{\ddot{\Phi}(t_*)} \right)^{1/2}$$

$$= \frac{4\mathcal{M}^{5/6}}{R\, \pi^{2/3}} \left( \frac{5}{6} \right)^{1/2} f^{-7/6}\, e^{i(2\pi f t_* - \Phi(t_*) - \pi/4)}$$

Can do a decent analysis of binary neutron star signals with this waveform

# Quasi-Elliptic Binary, Leading Order

$$h_+ = \frac{\mu M}{Ra(1-e^2)} \left( \left[ 2\cos(2\phi+2\psi) + \frac{5e}{2}\cos(\phi+2\psi) + \frac{e}{2}\cos(3\phi+2\psi) + e^2\cos(2\psi) \right] (1+\cos^2\iota) \right.$$
$$\left. + \left[ e\cos\phi + e^2 \right] \sin^2\iota \right)$$

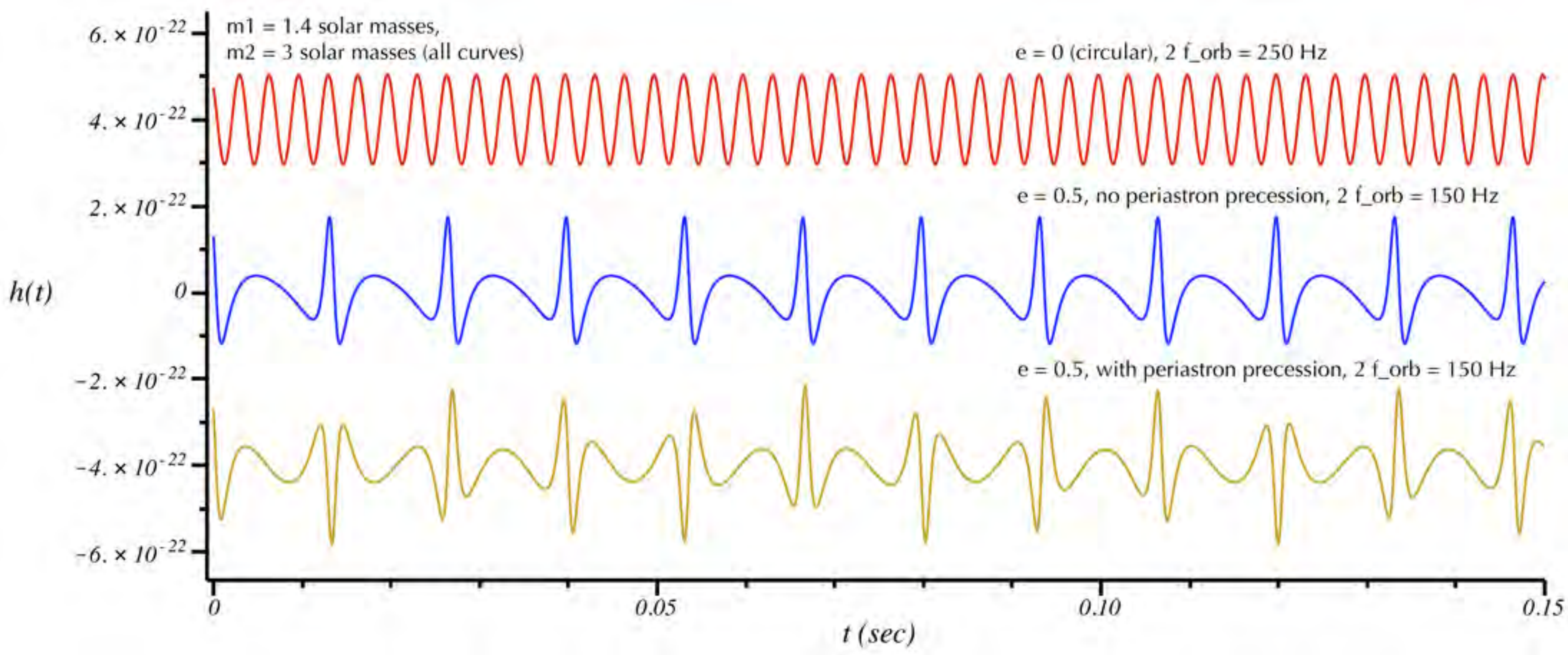
$$h_\times = \frac{\mu M}{Ra(1-e^2)} \left[ 4\sin(2\phi+2\psi) + 5e\sin(\phi+2\psi) + e\sin(3\phi+2\psi) + 2e^2\sin(2\psi) \right] \cos\iota$$

The waveforms depend on the first, second and third harmonics of the orbital phase, which itself is not simple harmonic:

$$\cos\phi = -e + \frac{2(1-e^2)}{e} \sum_{k=1}^{\infty} J_k(ke)\, \cos(k\omega t)$$

End result are waveforms with power spread over many harmonics

# Quasi-Elliptic Binary, Leading Order




0 PN

1 PN

Note: No radiation reaction included here

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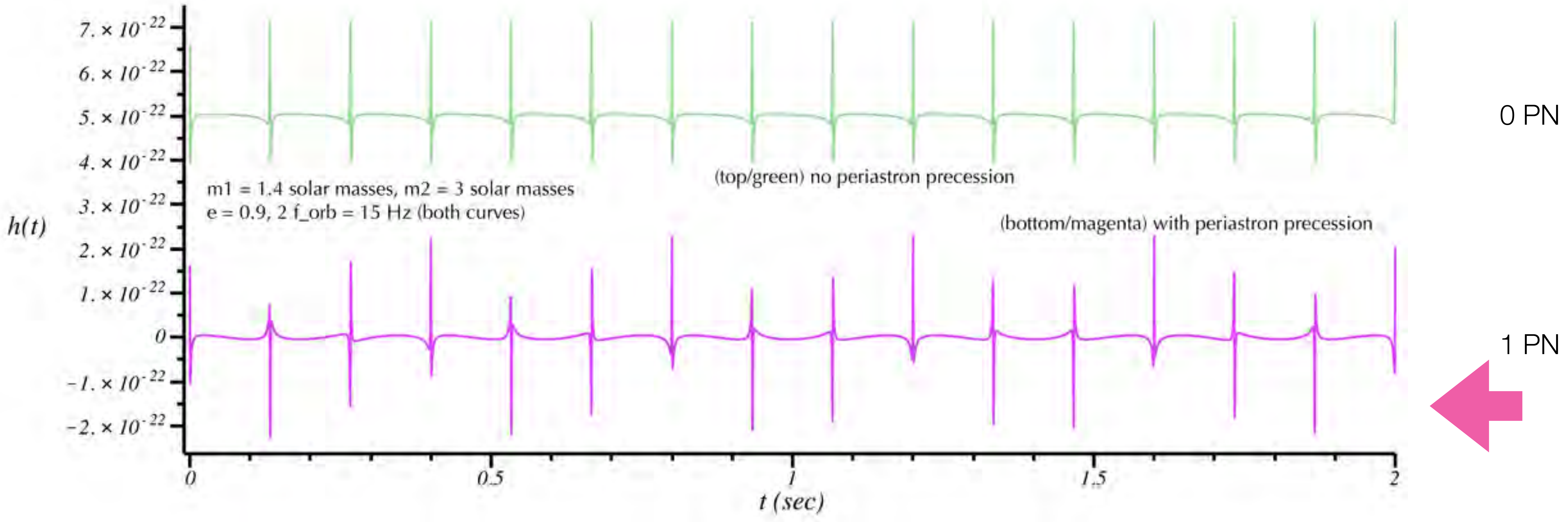
# Quasi-Elliptic Binary, Leading Order




Note: No radiation reaction included here

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# Quasi-Elliptic Binary, Radiation Reaction

$$\frac{dE}{dt} = -\frac{32\mu^2 M^3}{5a^5}\frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right) \qquad \frac{dL}{dt} = -\frac{32\mu^2 M^{5/2}}{5a^{7/2}}\frac{1}{(1-e^2)^2}\left(1+\frac{7}{8}e^2\right)$$

Combined with $\omega^2 a^3 = M$ $\quad E = -\frac{M\mu}{2a}$ $\quad L^2 = \mu^2 M a(1-e^2)$ $\quad e^2 = 1+\frac{2EL^2}{M^2\mu^3}$

Yields

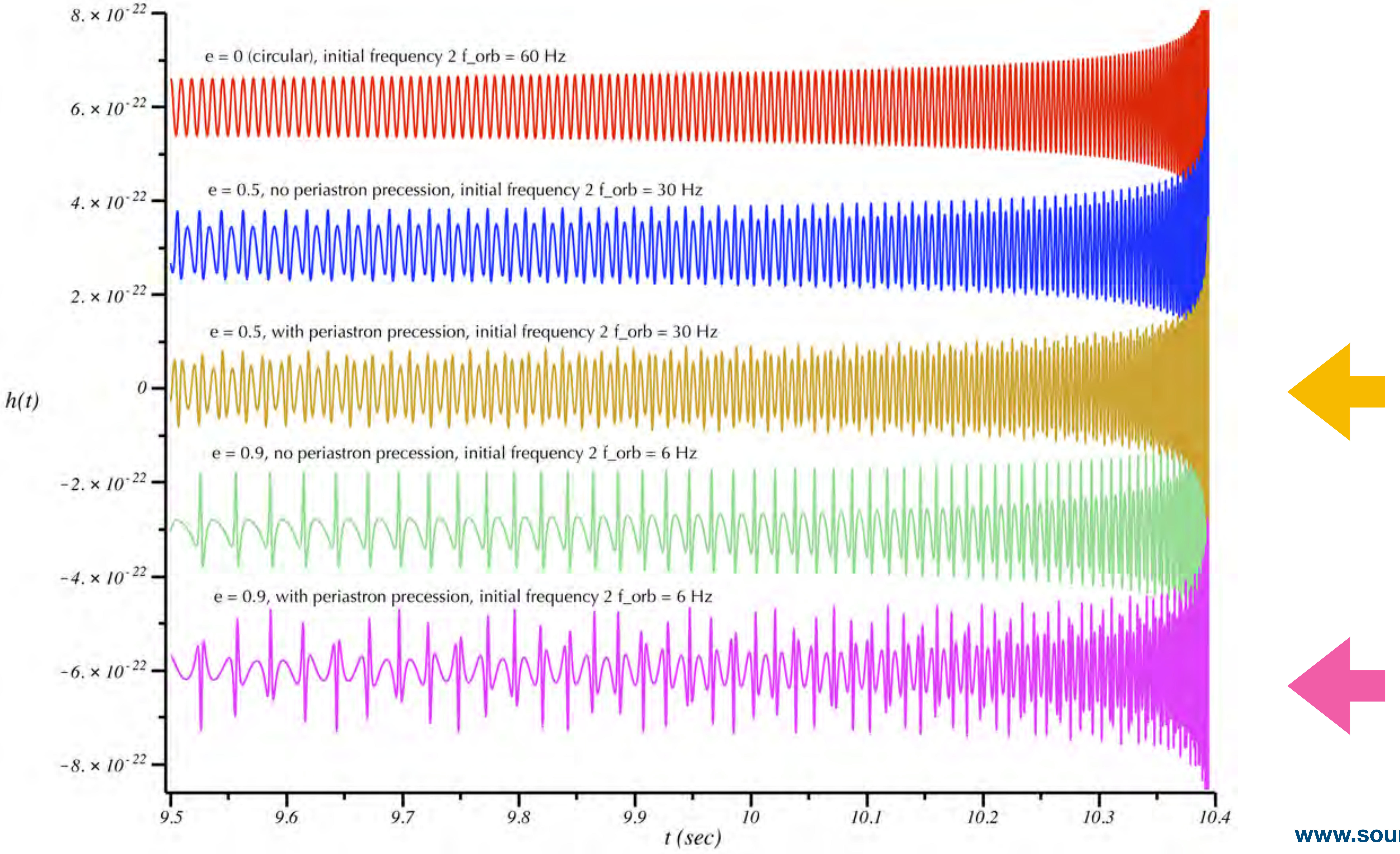
$$\dot{\omega} = \frac{96}{5}\mathcal{M}^{5/3}\,\omega^{11/3}\,\frac{1}{(1-e^2)}^{7/2}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)$$

$$\dot{e} = -\frac{304}{15}\mathcal{M}^{5/3}\,\omega^{8/3}\,\frac{e}{(1-e^2)^{5/2}}\left(1+\frac{121}{304}e^2\right)$$

$$\Rightarrow \quad \frac{\dot{e}}{e} \approx -\frac{19}{18}\frac{\dot{\omega}}{\omega}$$

Good rule of thumb - lose roughly a decade in eccentricity per decade in frequency

# Quasi-Elliptic Binary, Radiation Reaction

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# General Binary, Higher Orders

A variety of calculations techniques have been used to extend the waveforms to higher PN order: DIRE, matched asymptotic expansions, EFT…

$$h^{ij} = \frac{2\mu}{R}\left[h_0^{ij} + h_{0.5}^{ij} + h_1^{ij} + h_{1.5}^{ij} + \ldots\right]_{TT}$$

$$h_0^{ij} = 2\left[v^i v^j - \frac{M}{r} n^i n^j\right]$$

$$h_{0.5}^{ij} = \frac{\delta m}{M}\left\{3\frac{M}{r}\left[\dot{r} n^i n^j - 2n^{(i}v^{j)}\right](\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}) + \left[2v^i v^j - \frac{M}{r} n^i n^j\right](\hat{\mathbf{k}}\cdot\mathbf{v})\right\}$$

$$h_1^{ij} = \frac{1}{3}(1-3\eta)\left\{4\frac{M}{r}\left[3\dot{r} n^i n^j - 8n^{(i}v^{j)}\right](\hat{\mathbf{k}}\cdot\hat{\mathbf{n}})(\hat{\mathbf{k}}\cdot\mathbf{v}) + 2\left[3v^i v^j - \frac{M}{r} n^i n^j\right](\hat{\mathbf{k}}\cdot\mathbf{v})^2 + \frac{M}{r}\left[(3v^2 - 15\dot{r}^2 + 7\frac{M}{r})n^i n^j + 30\dot{r} n^{(i}v^{j)} - 14 v^i v^j\right](\hat{\mathbf{k}}\cdot\hat{\mathbf{n}})^2\right\} + \frac{4}{3}\frac{M}{r}\dot{r}(5+3\eta)n^{(i}v^{j)} + \left[(1-3\eta)v^2 - \frac{2}{3}(2-3\eta)\frac{M}{r}\right]v^i v^j + \frac{M}{r}\left[(1-3\eta)\dot{r}^2 - \frac{1}{3}(10+3\eta)v^2 + \frac{29}{3}\frac{M}{r}\right]n^i n^j,$$

# Quasi-Circular Binary, Higher Orders

Energy:

$$E = -\frac{1}{2}\frac{\mu M}{r}\left\{1 - \frac{1}{4}(7-\eta)\left(\frac{M}{r}\right) + \sum_{i=1,2}\left[\chi_i(\hat{\mathbf{L}}_{\mathbf{N}}\cdot\hat{\mathbf{s}}_{\mathbf{i}})(2\frac{m_i^2}{M^2}+\eta)\right]\left(\frac{M}{r}\right)^{3/2} - \left[\frac{1}{8}(7-49\eta-\eta^2) - \frac{1}{2}\eta\chi_1\chi_2\left[(\hat{\mathbf{s}}_{\mathbf{1}}\cdot\hat{\mathbf{s}}_{\mathbf{2}}) - 3(\hat{\mathbf{L}}_{\mathbf{N}}\cdot\hat{\mathbf{s}}_{\mathbf{1}})(\hat{\mathbf{L}}_{\mathbf{N}}\cdot\hat{\mathbf{s}}_{\mathbf{2}})\right]\right]\left(\frac{M}{r}\right)^2\right\}$$

Energy Flux:

$$\frac{dE}{dt} = -\frac{32}{5}\eta^2\left(\frac{M}{r}\right)^5\left\{1 - \frac{1}{336}(2927+420\eta)\left(\frac{M}{r}\right) - \left[\frac{1}{12}\sum_{i=1,2}\left[\chi_i(\hat{\mathbf{L}}_{\mathbf{N}}\cdot\hat{\mathbf{s}}_{\mathbf{i}})(73\frac{m_i^2}{M^2}+75\eta)\right] - 4\pi\right]\left(\frac{M}{r}\right)^{3/2} - \frac{1}{48}\eta\chi_1\chi_2\left[223(\hat{\mathbf{s}}_{\mathbf{1}}\cdot\hat{\mathbf{s}}_{\mathbf{2}}) - 649(\hat{\mathbf{L}}_{\mathbf{N}}\cdot\hat{\mathbf{s}}_{\mathbf{1}})(\hat{\mathbf{L}}_{\mathbf{N}}\cdot\hat{\mathbf{s}}_{\mathbf{2}})\right]\left(\frac{M}{r}\right)^2\right\}$$

# Quasi-Circular Binary, Orbital Decay

Corrected Kepler:

$$\omega^2 r^3 = M\left\{1-(3-\eta)\left(\frac{M}{r}\right) - \sum_{i=1,2}\left[\chi_i(\hat{\mathbf{L}}_{\mathbf{N}}\cdot\hat{\mathbf{s}}_{\mathbf{i}})(2\frac{m_i^2}{M^2}+3\eta)\right]\left(\frac{M}{r}\right)^{3/2} + \left[(6+\frac{41}{4}\eta+\eta^2) - \frac{3}{2}\eta\chi_1\chi_2\left[(\hat{\mathbf{s}}_{\mathbf{1}}\cdot\hat{\mathbf{s}}_{\mathbf{2}}) - 3(\hat{\mathbf{L}}_{\mathbf{N}}\cdot\hat{\mathbf{s}}_{\mathbf{1}})(\hat{\mathbf{L}}_{\mathbf{N}}\cdot\hat{\mathbf{s}}_{\mathbf{2}})\right]\right]\left(\frac{M}{r}\right)^2\right\}$$

Combining E, dE/dt and corrected Kepler get

$$\dot{\omega} = \frac{96}{5}\mathcal{M}^{5/3}\omega^{11/3}\left\{1-\frac{1}{336}(743+924\eta)(M\omega)^{2/3} - \left[\frac{1}{12}\sum_{i=1,2}\left[\chi_i(\hat{\mathbf{L}}_{\mathbf{N}}\cdot\hat{\mathbf{s}}_{\mathbf{i}})(113\frac{m_i^2}{M^2}+75\eta)\right] - 4\pi\right](M\omega) - \frac{1}{48}\eta\chi_1\chi_2\left[247(\hat{\mathbf{s}}_{\mathbf{1}}\cdot\hat{\mathbf{s}}_{\mathbf{2}}) - 721(\hat{\mathbf{L}}_{\mathbf{N}}\cdot\hat{\mathbf{s}}_{\mathbf{1}})(\hat{\mathbf{L}}_{\mathbf{N}}\cdot\hat{\mathbf{s}}_{\mathbf{2}})\right](M\omega)^{4/3}\right\}$$

# Quasi-Circular Binary, Orbital Decay

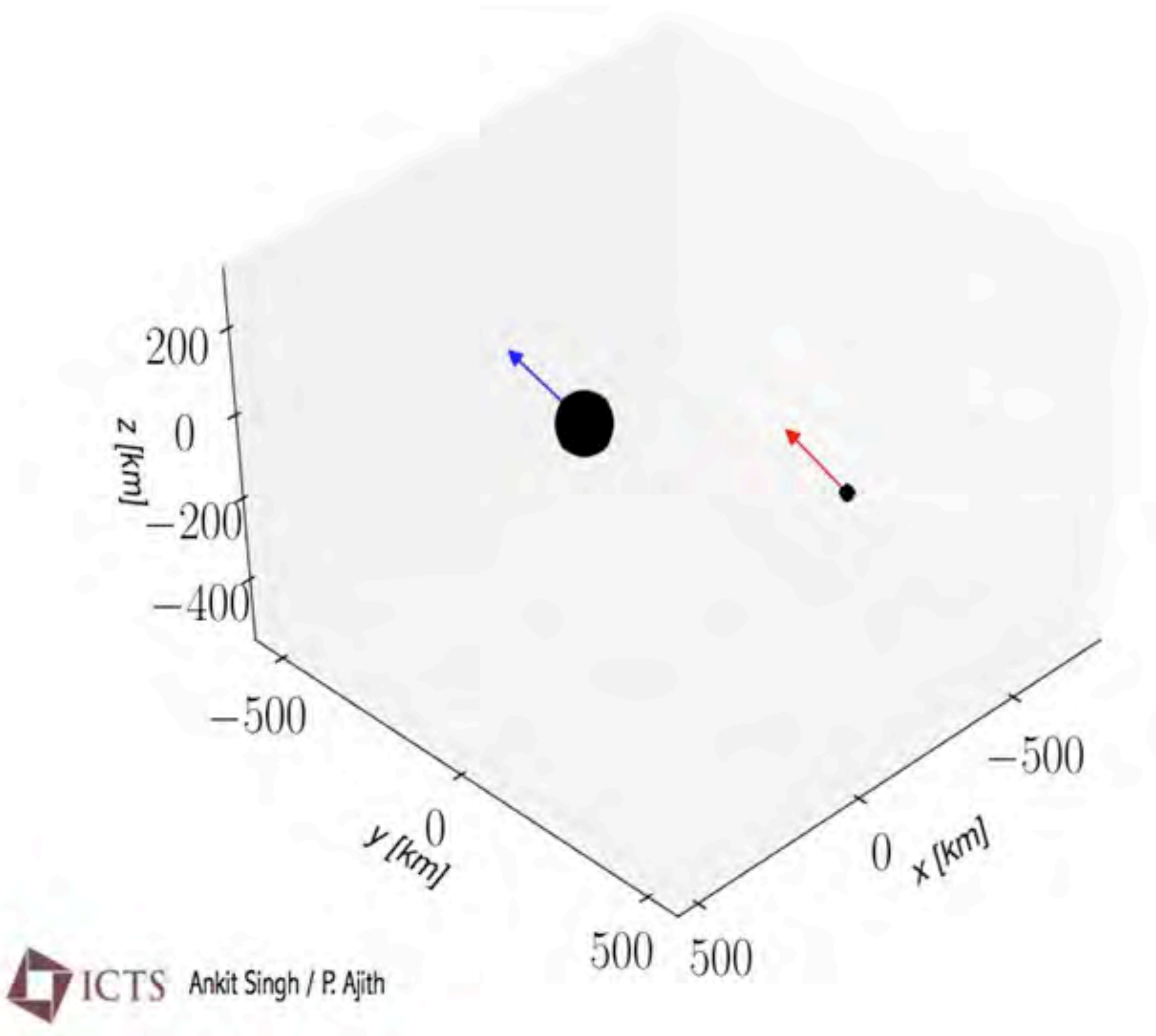
$$t(f) = t_c - \frac{5}{256}\mathcal{M}(\pi\mathcal{M}f)^{-8/3}\left[1 + \frac{4}{3}\left(\frac{743}{336} + \frac{11}{4}\eta\right)(\pi M f)^{2/3} - \frac{8}{5}(4\pi - \beta)(\pi M f) + 2\left(\frac{3058673}{1016064} + \frac{5429}{1008}\eta + \frac{617}{144}\eta^2 - \sigma\right)(\pi M f)^{4/3}\right]$$

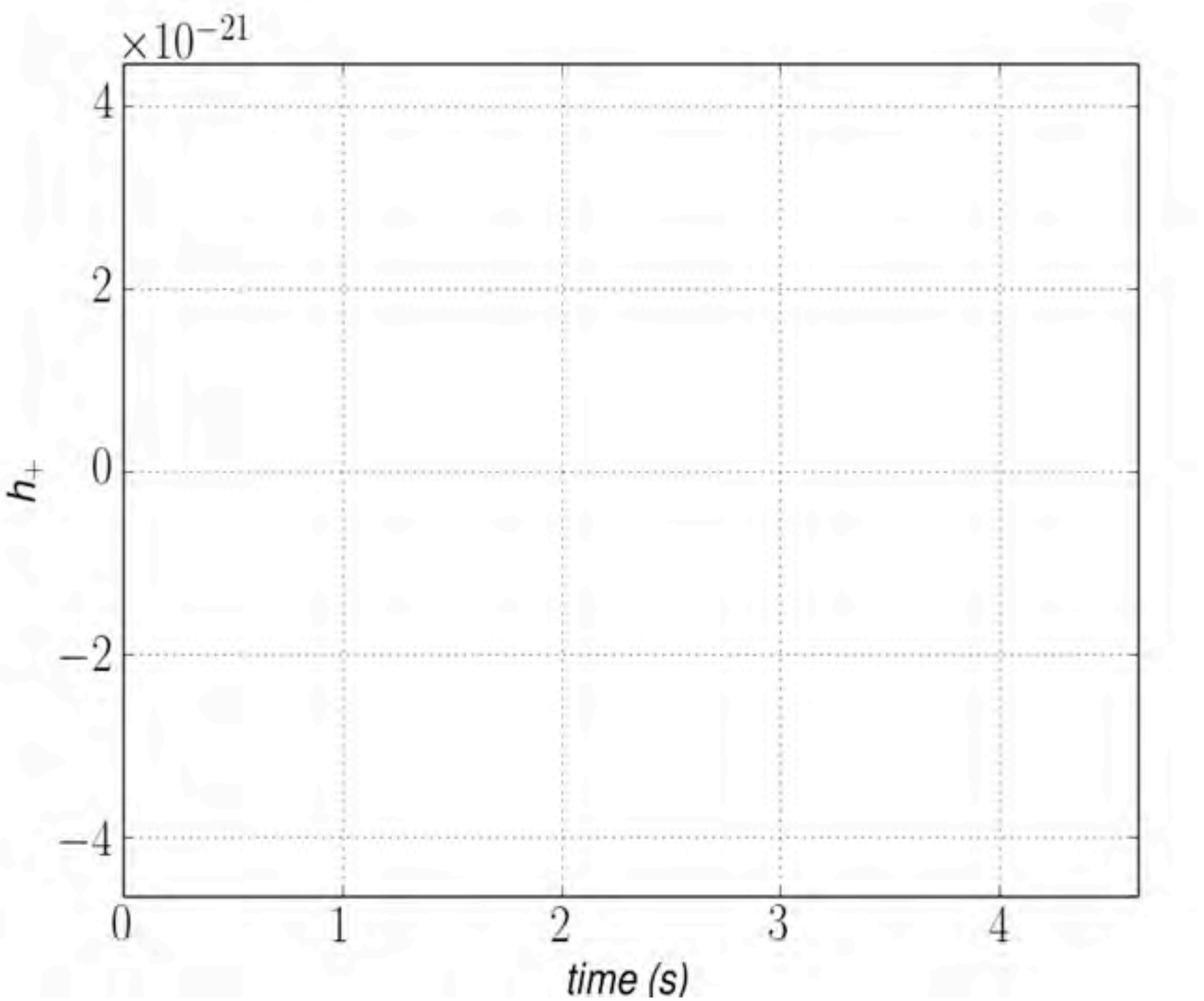
$$\Phi(f) = \Phi[t(f)] = \Phi_c - \frac{1}{16}(\pi\mathcal{M}f)^{-5/3}\left[1 + \frac{5}{3}\left(\frac{743}{336} + \frac{11}{4}\eta\right)(\pi M f)^{2/3} - \frac{5}{2}(4\pi - \beta)(\pi M f) + 5\left(\frac{3058673}{1016064} + \frac{5429}{1008}\eta + \frac{617}{144}\eta^2 - \sigma\right)(\pi M f)^{4/3}\right]$$

$$\beta = \frac{1}{12}\sum_{i=1}^{2}\left[113\left(\frac{m_i}{M}\right)^2 + 75\frac{\mu}{M}\right]\frac{\hat{\mathbf{L}}\cdot\mathbf{S}_i}{m_i^2}$$

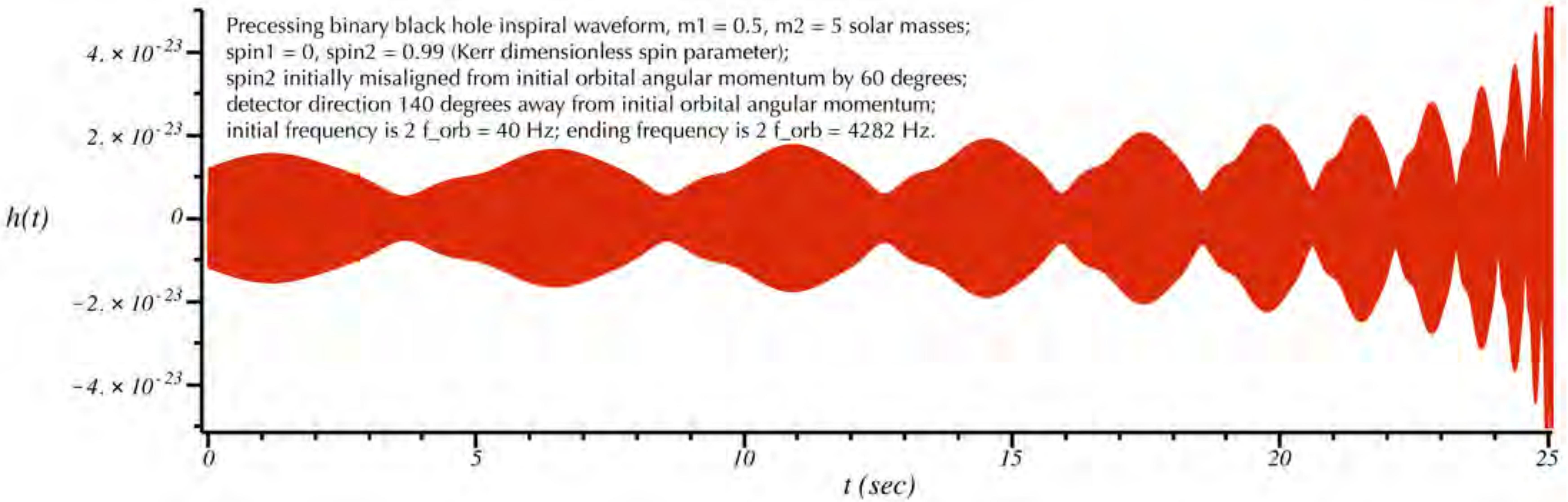
$$\sigma = \frac{\mu}{48M(m_1^2 m_2^2)}[721(\hat{\mathbf{L}}\cdot\mathbf{S}_1)(\hat{\mathbf{L}}\cdot\mathbf{S}_2) - 247(\mathbf{S}_1\cdot\mathbf{S}_2)]$$

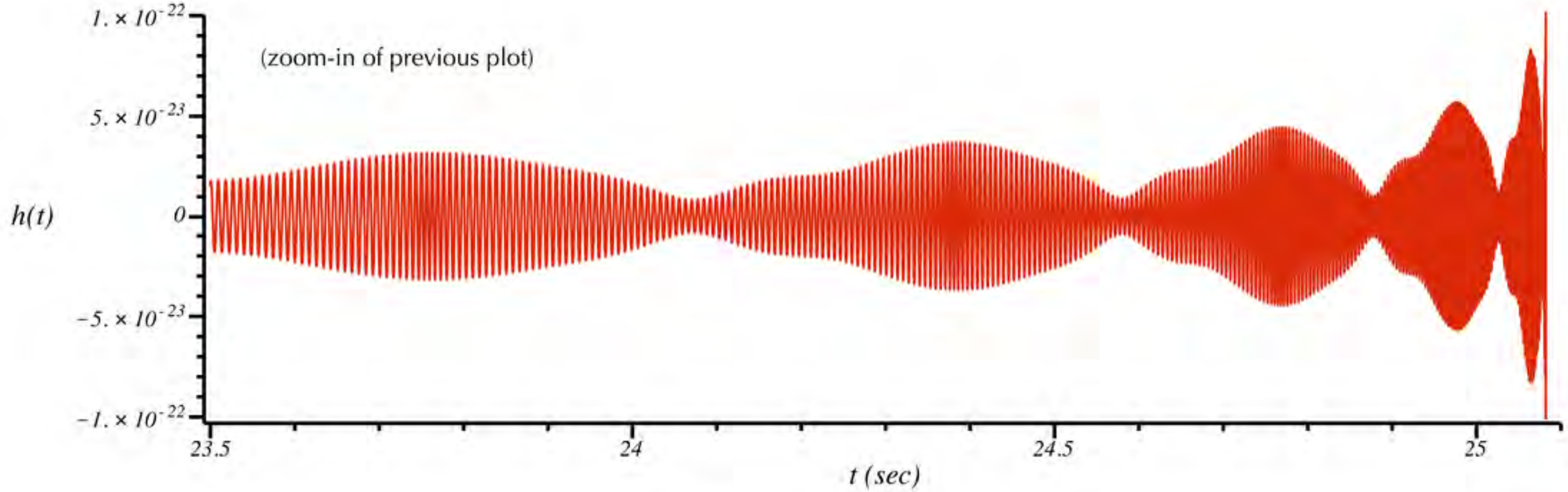
# Quasi-Circular Binary, Spin Precession



# Quasi-Circular Binary, Spin Precession

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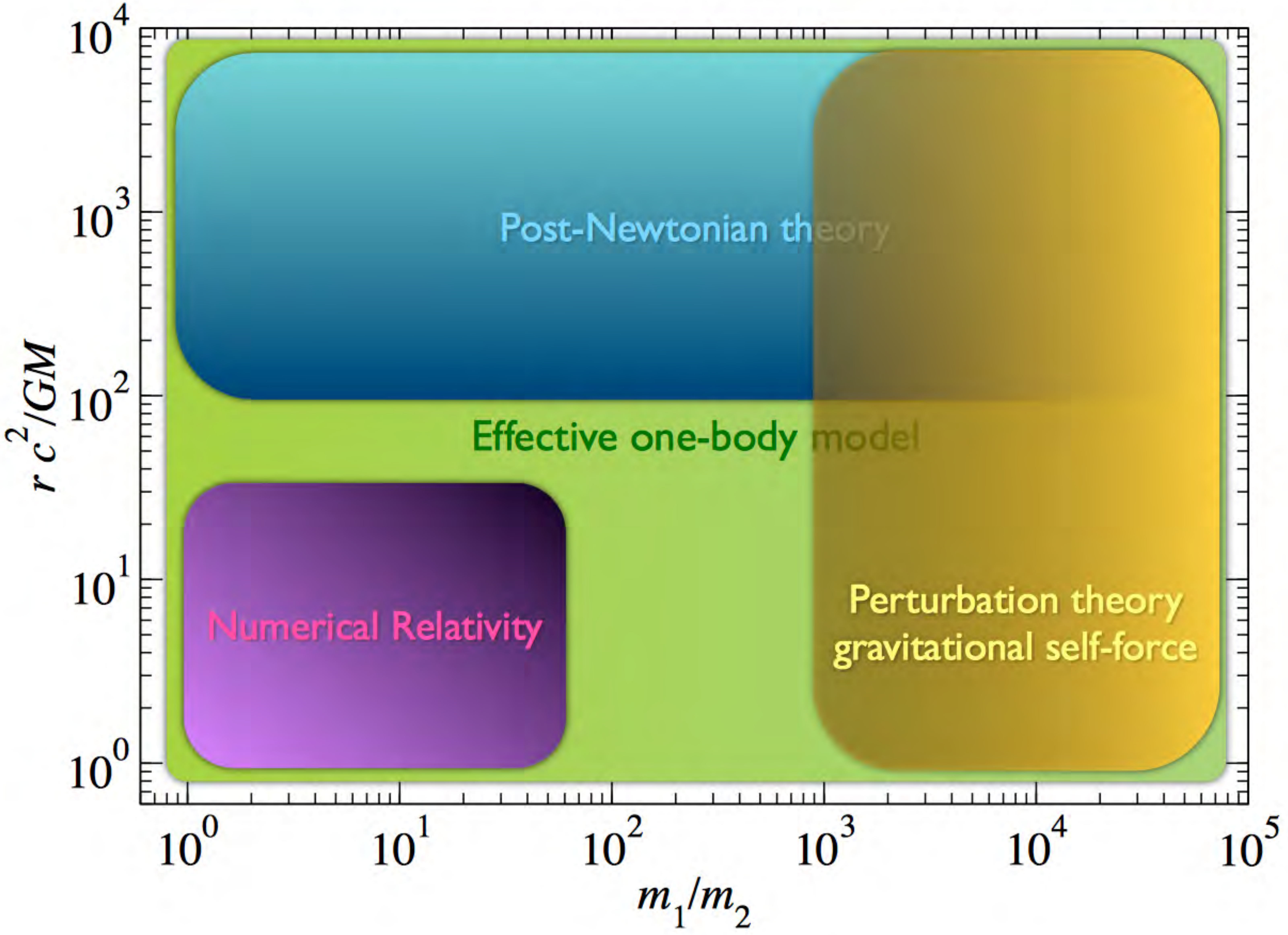
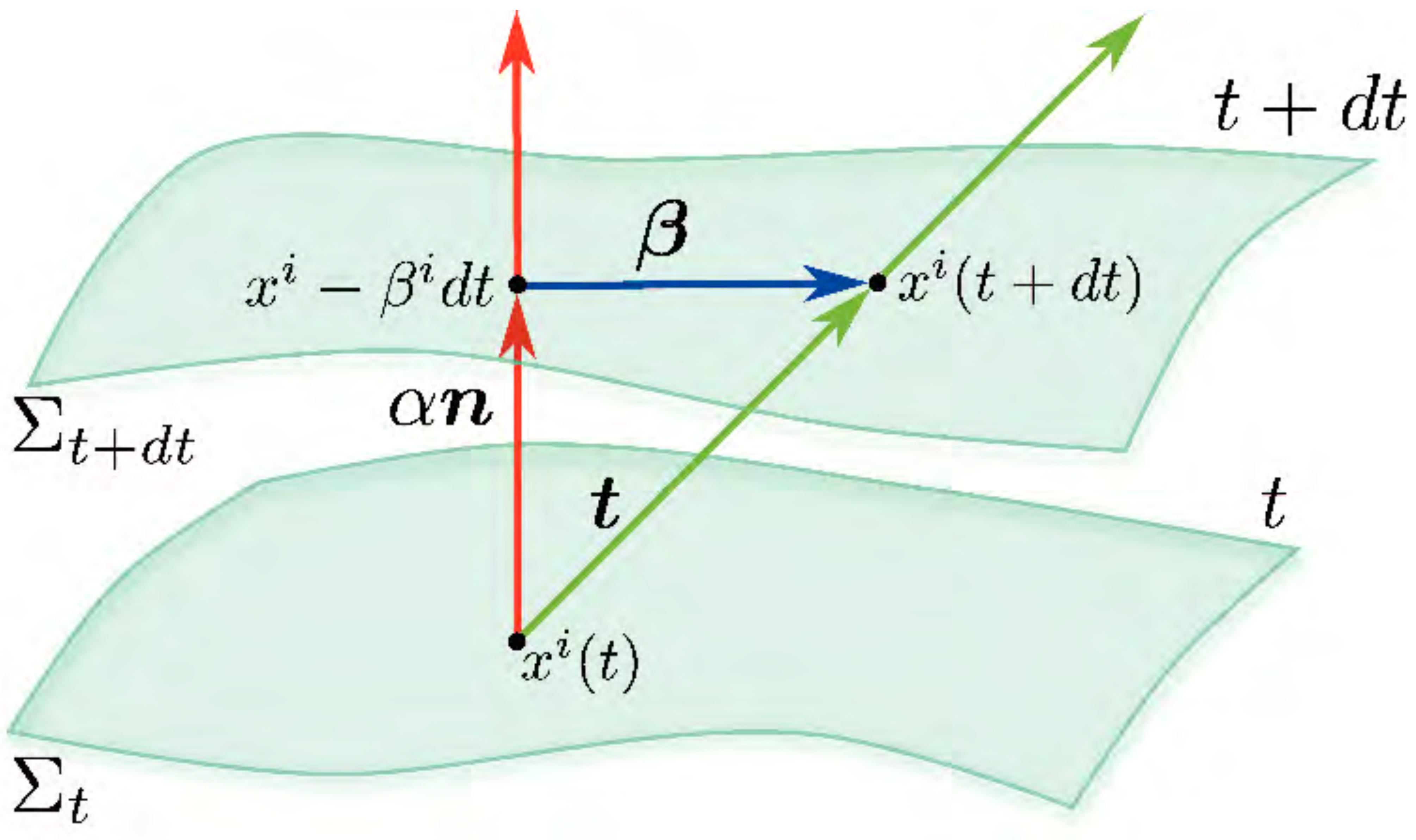
# Beyond Post Newtonian



# Numerical Relativity

(3+1) decomposition




$\gamma_{ij}$ metric on spatial hyper surfaces

$\alpha$ lapse

$\beta^i$ shift

$$g_{tt} = -\alpha^2 + \beta_i \beta^i \qquad g_{ti} = \gamma_{ij}\beta^j \qquad g_{ij} = \gamma_{ij}$$

Evolution equations:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i$$

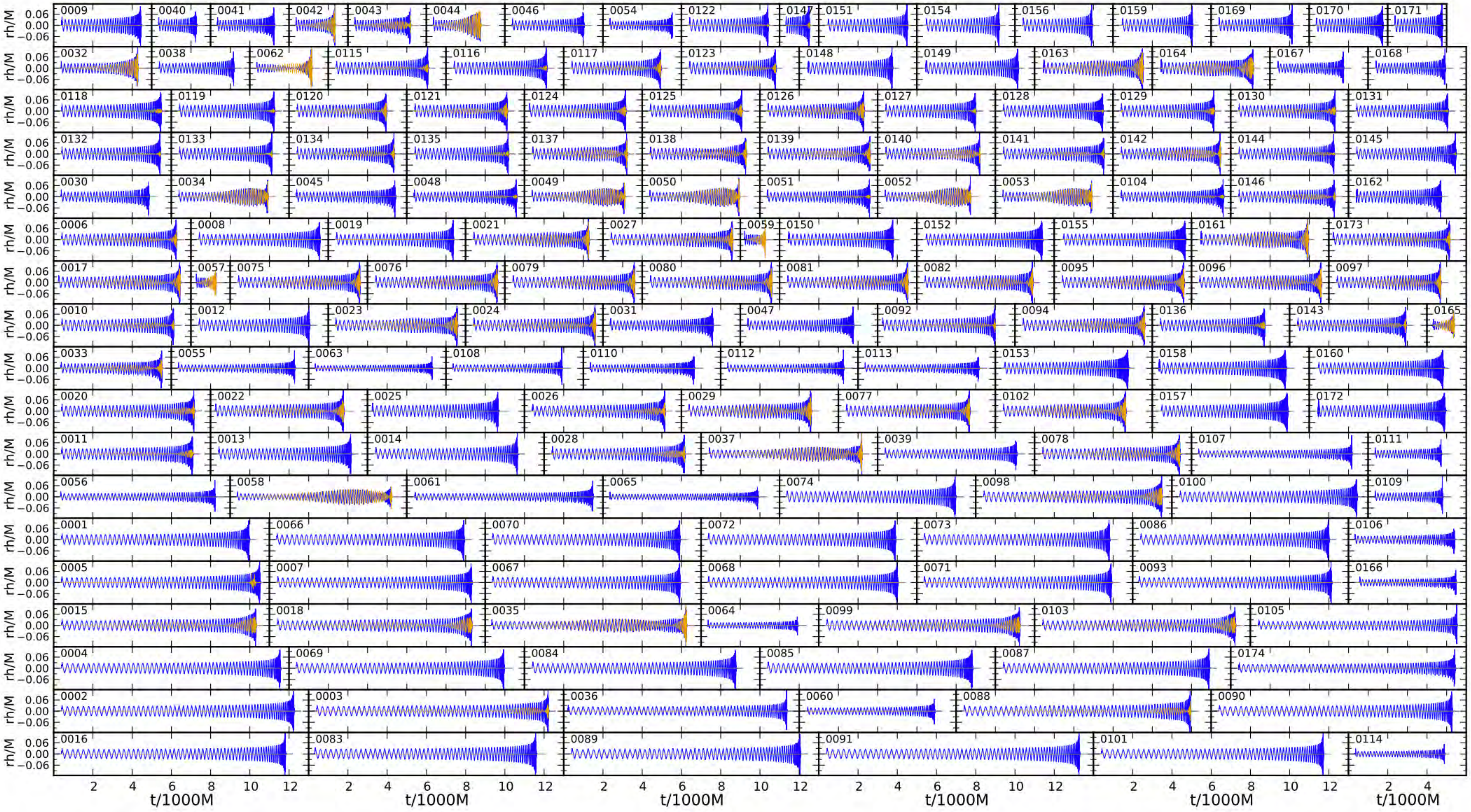
$$\partial_t K_{ij} = -\nabla_j \nabla_i \alpha + \alpha \left( R_{ij} + K K_{ij} - 2 K_{im} K^m_j \right) + \beta^m \nabla_m K_{ij} + K_{im} \nabla_j \beta^m + K_{mj} \nabla_i \beta^m$$

Constraint equations:

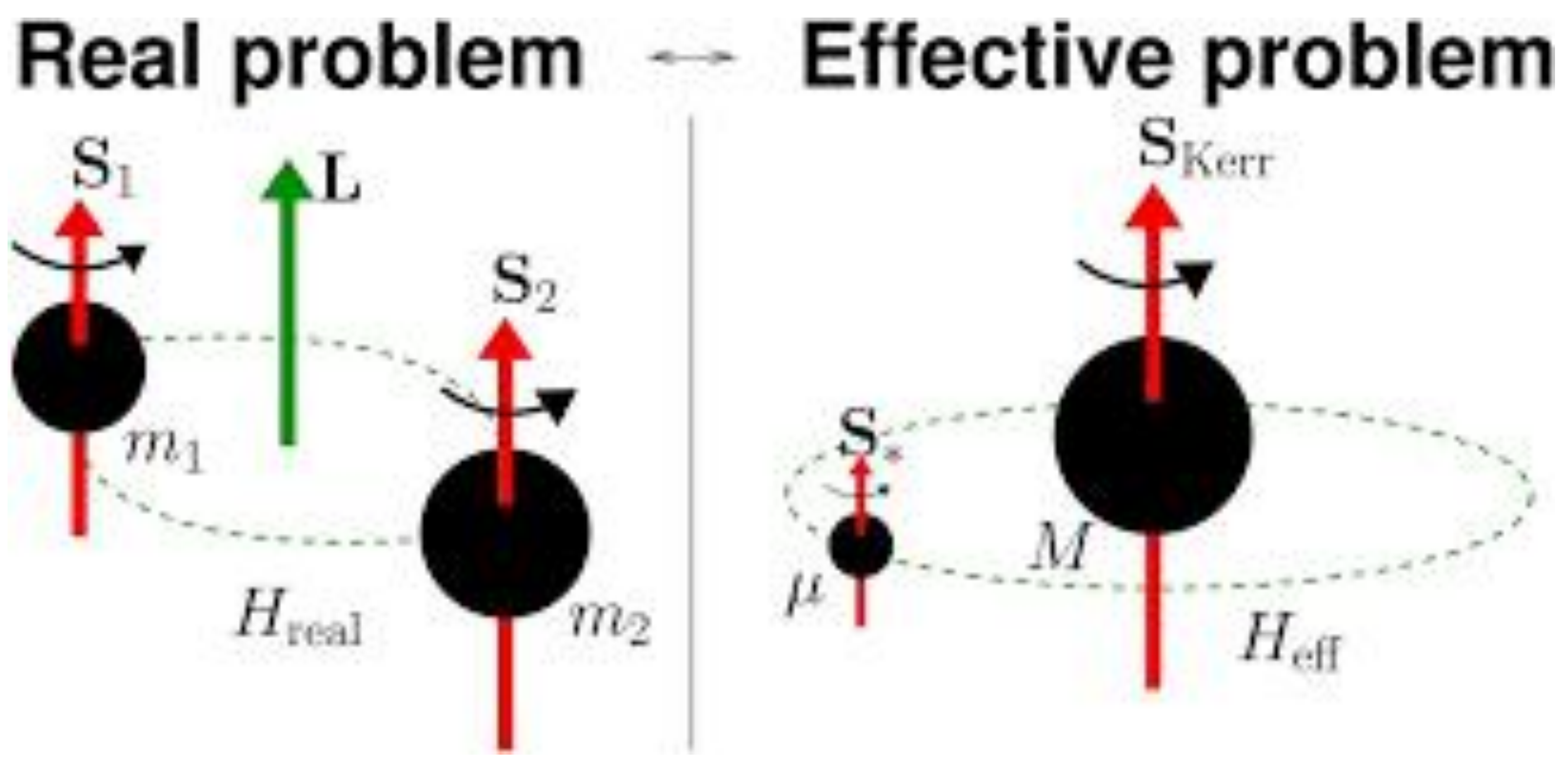
$$R + K^2 = K_{ij} K^{ij}$$

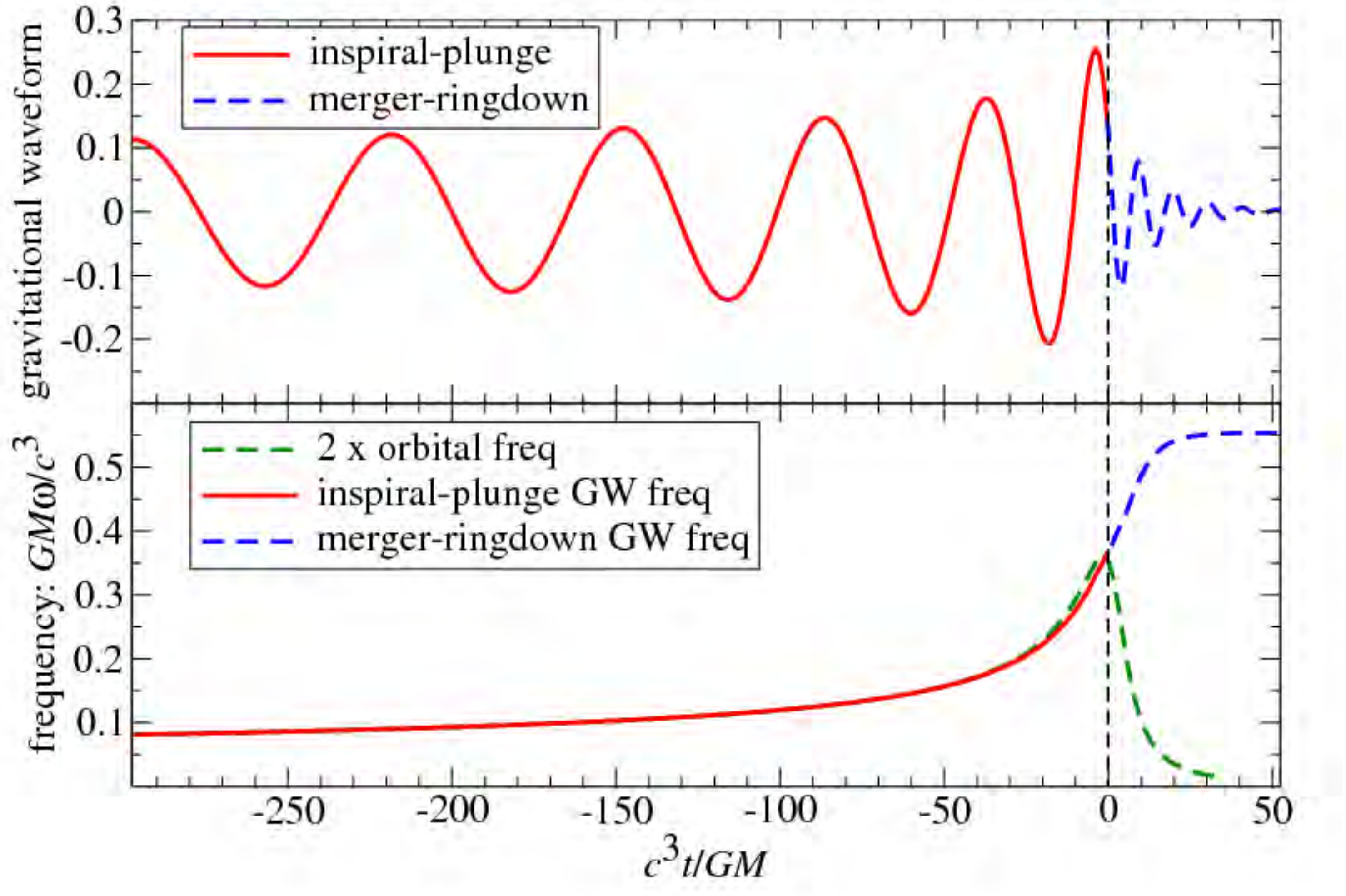
$$\nabla_j K^j_i = \nabla_i K$$

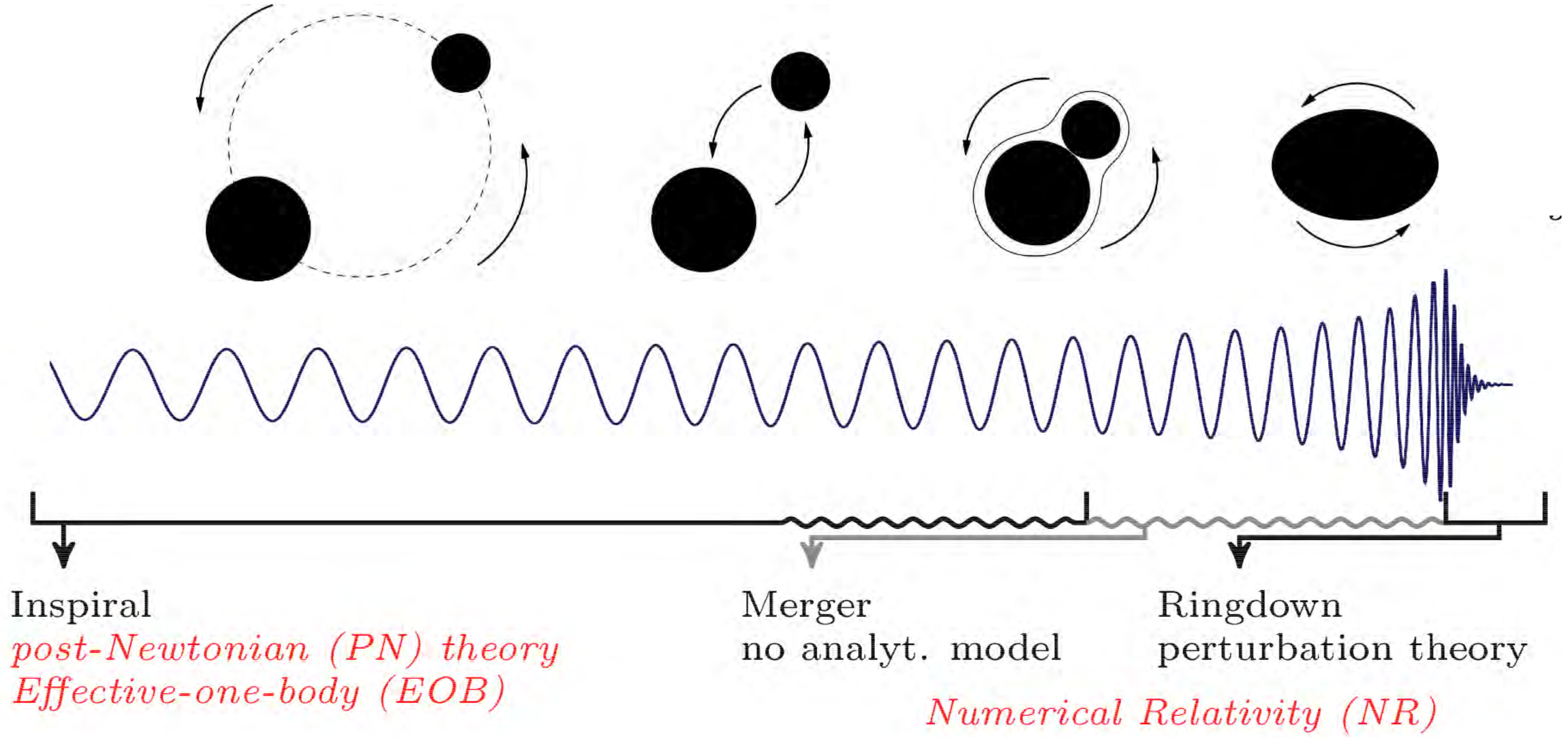
# Numerical Relativity



# Effective One-body



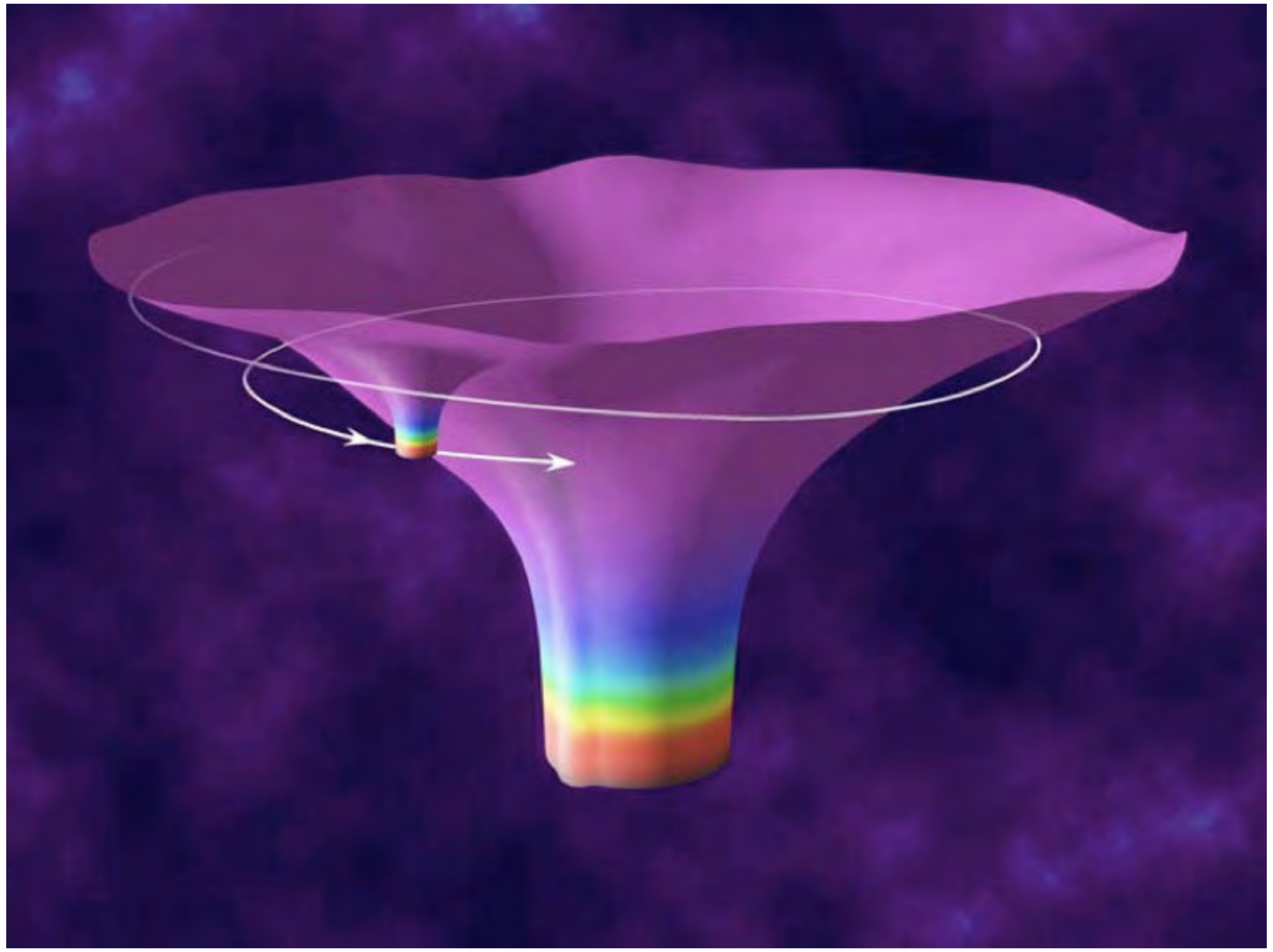



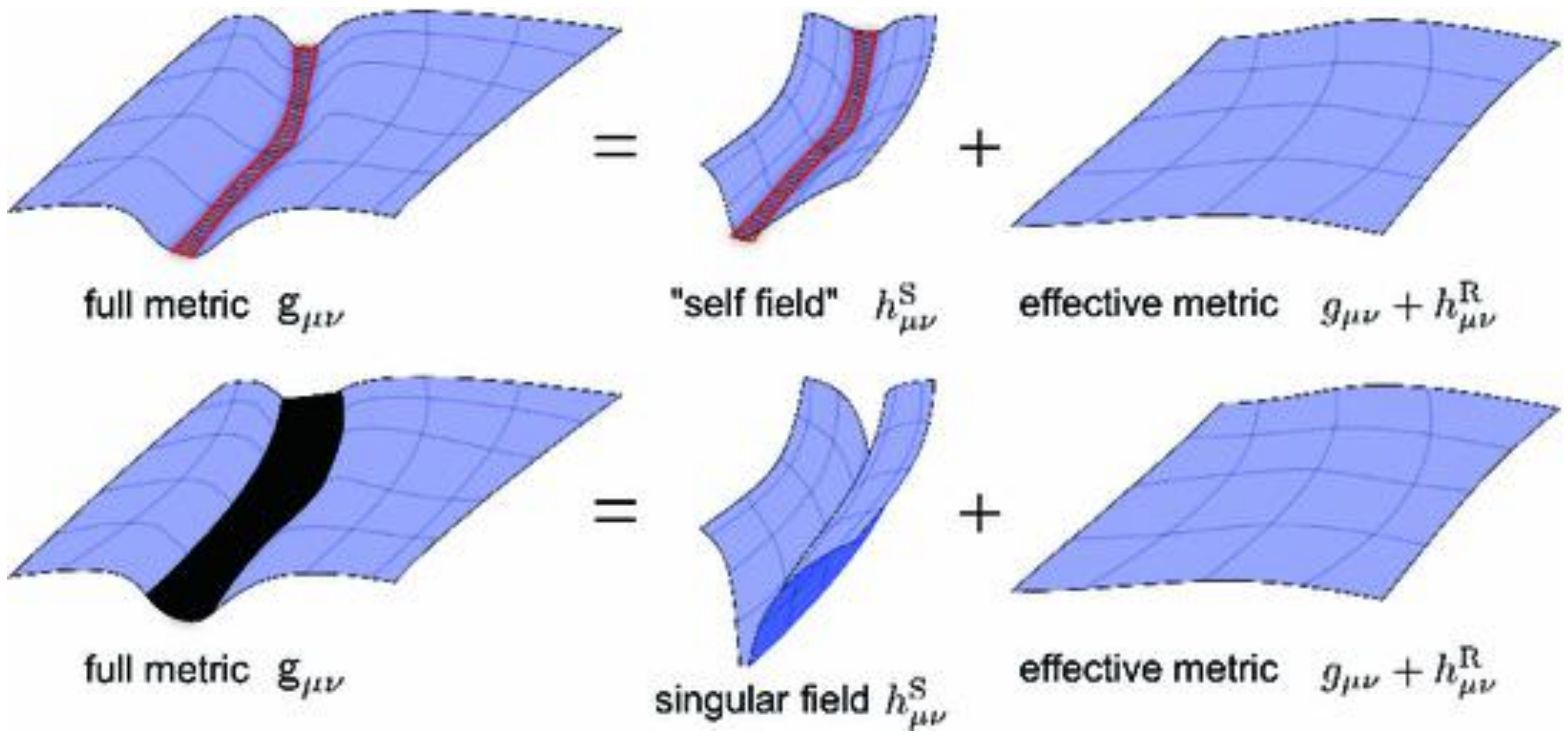




Re-summation of the PN expansion attached to quasi-normal modes of a perturbed black hole

# Self-force, EMRI

[Review: A. Pound, Equations of Motion in Relativistic Gravity (Book), 399 - 486, Springer]





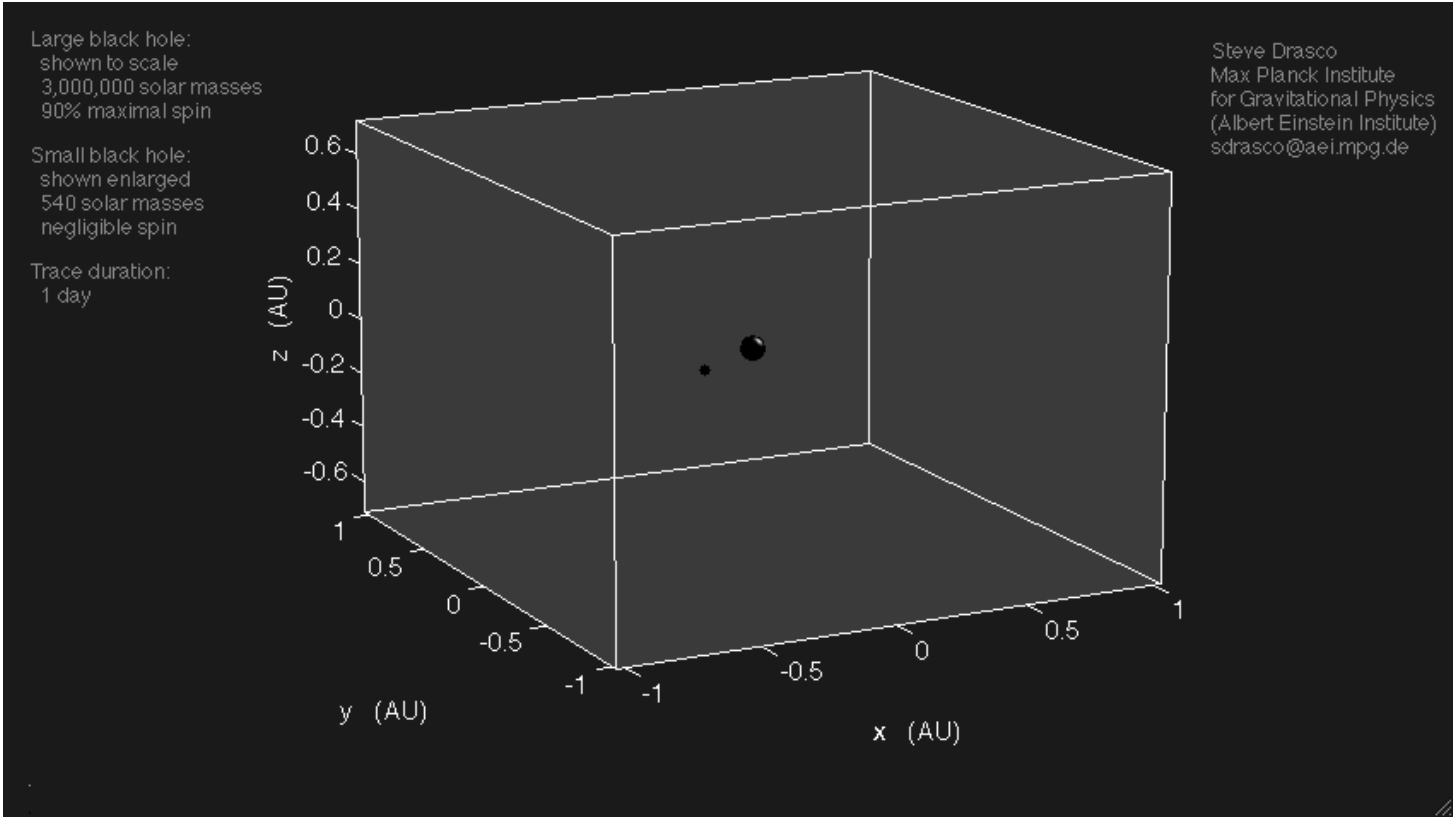

MiSaTaQuWa Equation

$$u^{\mu}\nabla_{\mu}u^{\nu} = -(g^{\nu\kappa} + u^{\nu}u^{\kappa})(\nabla_{\alpha}h^{\text{tail}}_{\kappa\gamma} - \frac{1}{2}\nabla_{\kappa}h^{\text{tail}}_{\gamma\alpha})u^{\gamma}u^{\alpha}$$

Detweiler-Whiting Equation

$$u^{\mu}\nabla_{\mu}u^{\nu} = -(g^{\nu\kappa} + u^{\nu}u^{\kappa})(\nabla_{\alpha}h^{\text{Reg}}_{\kappa\gamma} - \frac{1}{2}\nabla_{\kappa}h^{\text{Reg}}_{\gamma\alpha})u^{\gamma}u^{\alpha}$$

# Self-force, EMRI



# Physics and Sources of Gravitational Waves

Neil J. Cornish

eXtreme Gravity Institute

MONTANA STATE UNIVERSITY

# Part 2. Gravitational Wave Sources

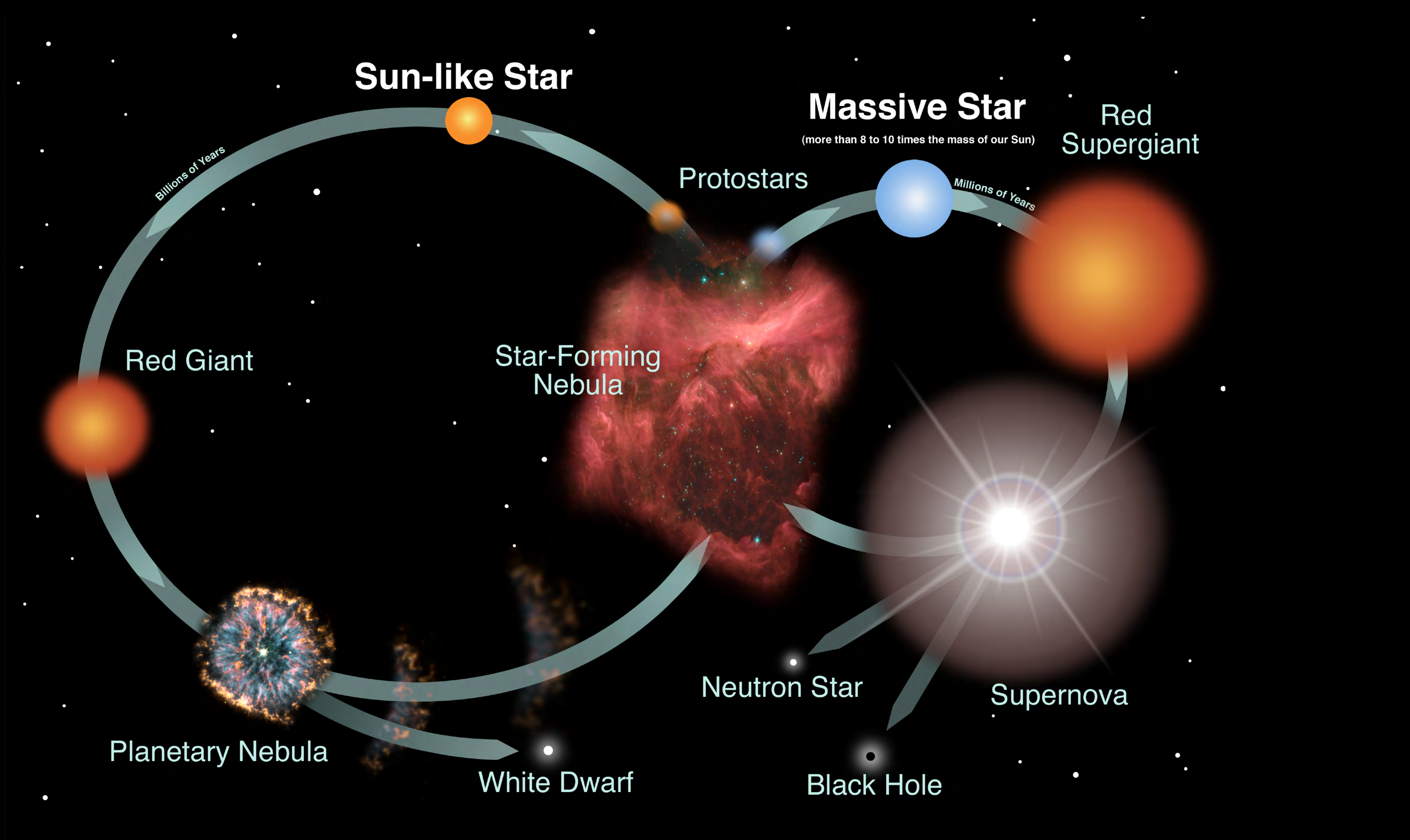
# Recipe for Making Gravitational Waves

**Ingredients**:

- Large amount of mass/energy (any type will do).

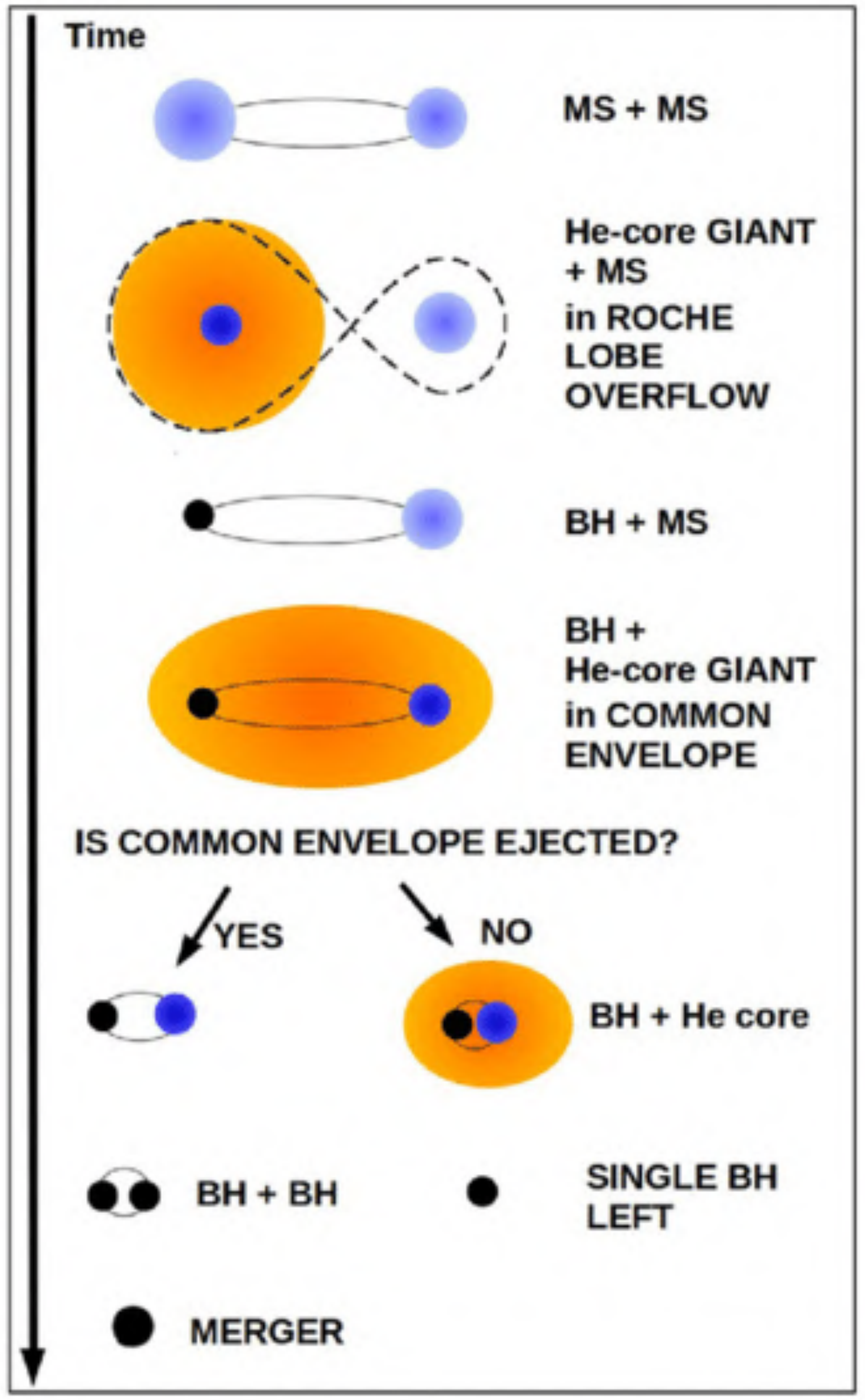
**Directions:**

- Squish into a small, lumpy blob. (two blobs work better)
- Shake or stir vigorously.


Sun-like Star
Billions of Years
Massive Star
(more than 8 to 10 times the mass of our Sun)
Red Supergiant
Protostars
Millions of Years
Red Giant
Star-Forming Nebula
Neutron Star
Supernova
Planetary Nebula
White Dwarf
Black Hole

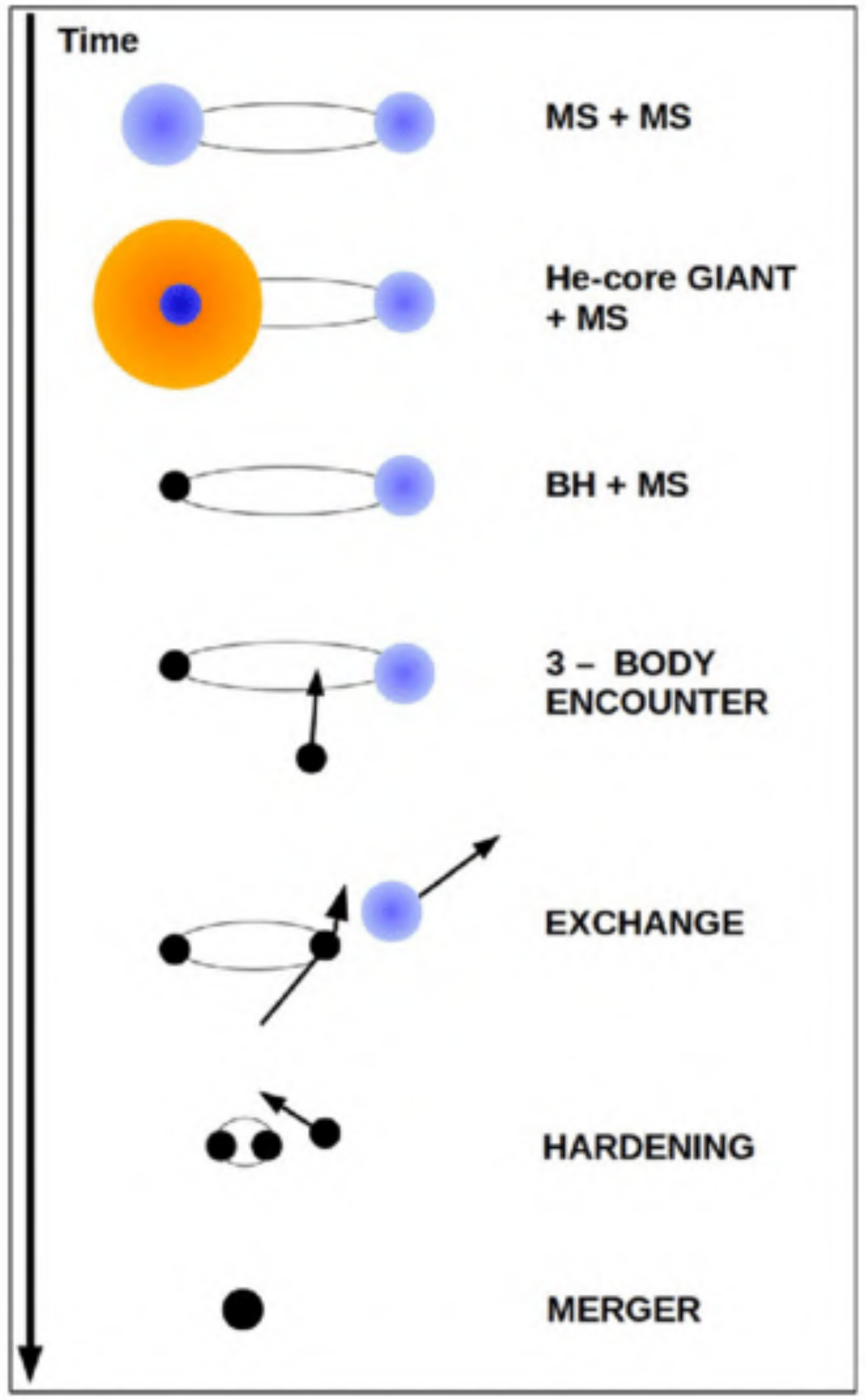
LIGO-Virgo-KAGRA Black Holes
LIGO-Virgo-KAGRA Neutron Stars
Solar Masses
200
100
50
20
10
5
2
1

# Stellar remnant binaries - formation channels for BHBs



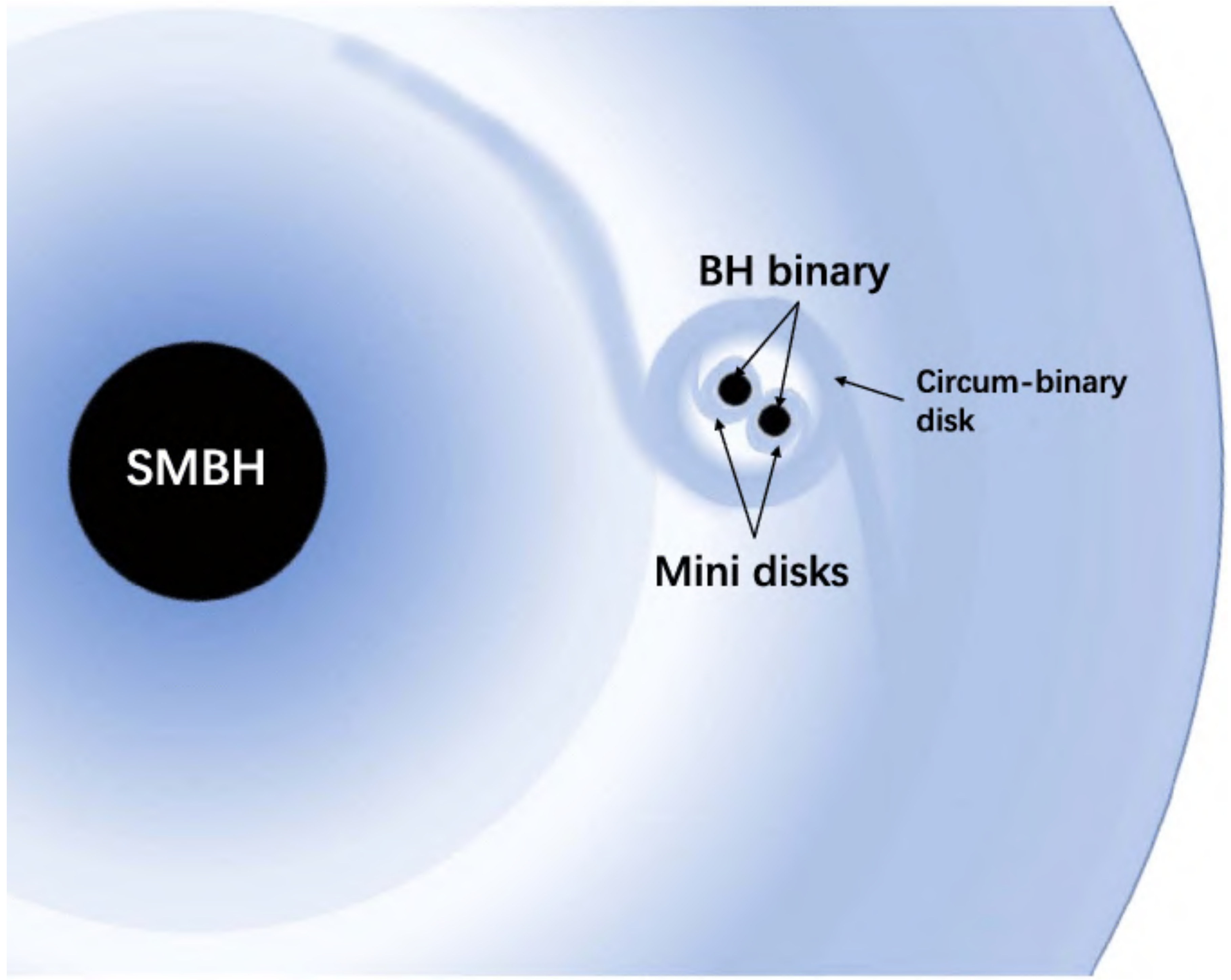

Field

**Circular, low, aligned spins?**




Cluster

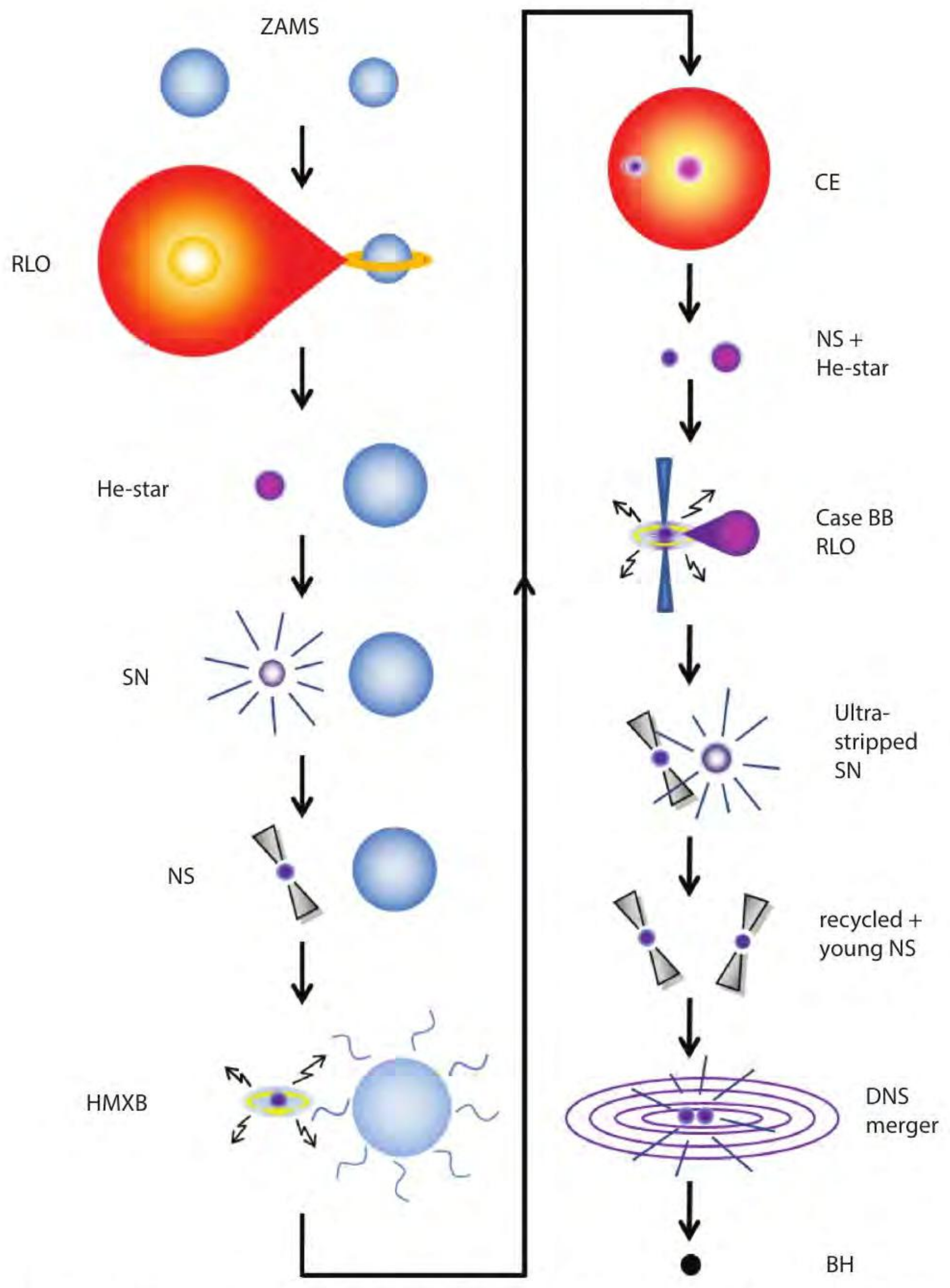
**Circular, moderate, mis-aligned spins?**



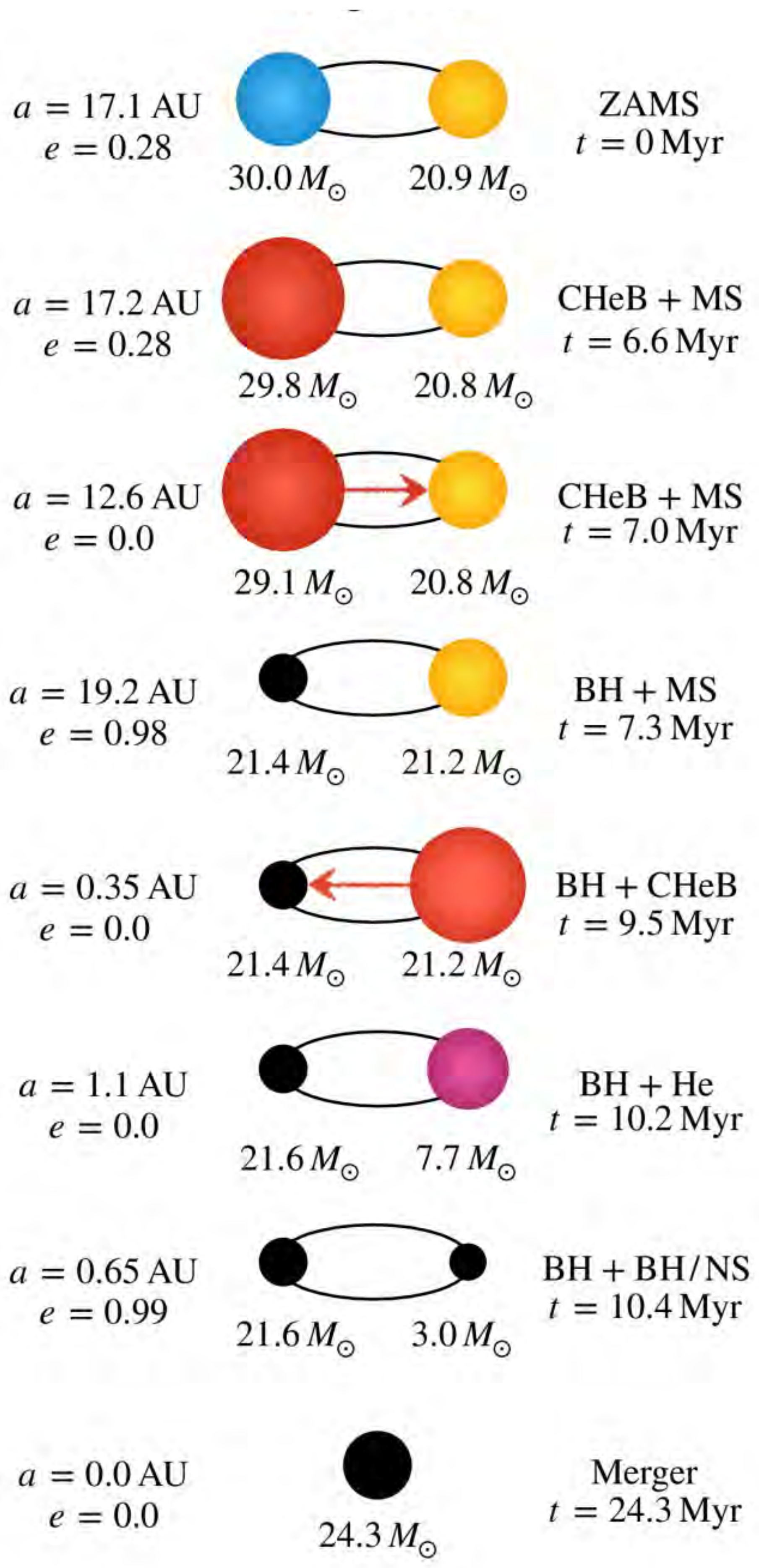

Disk

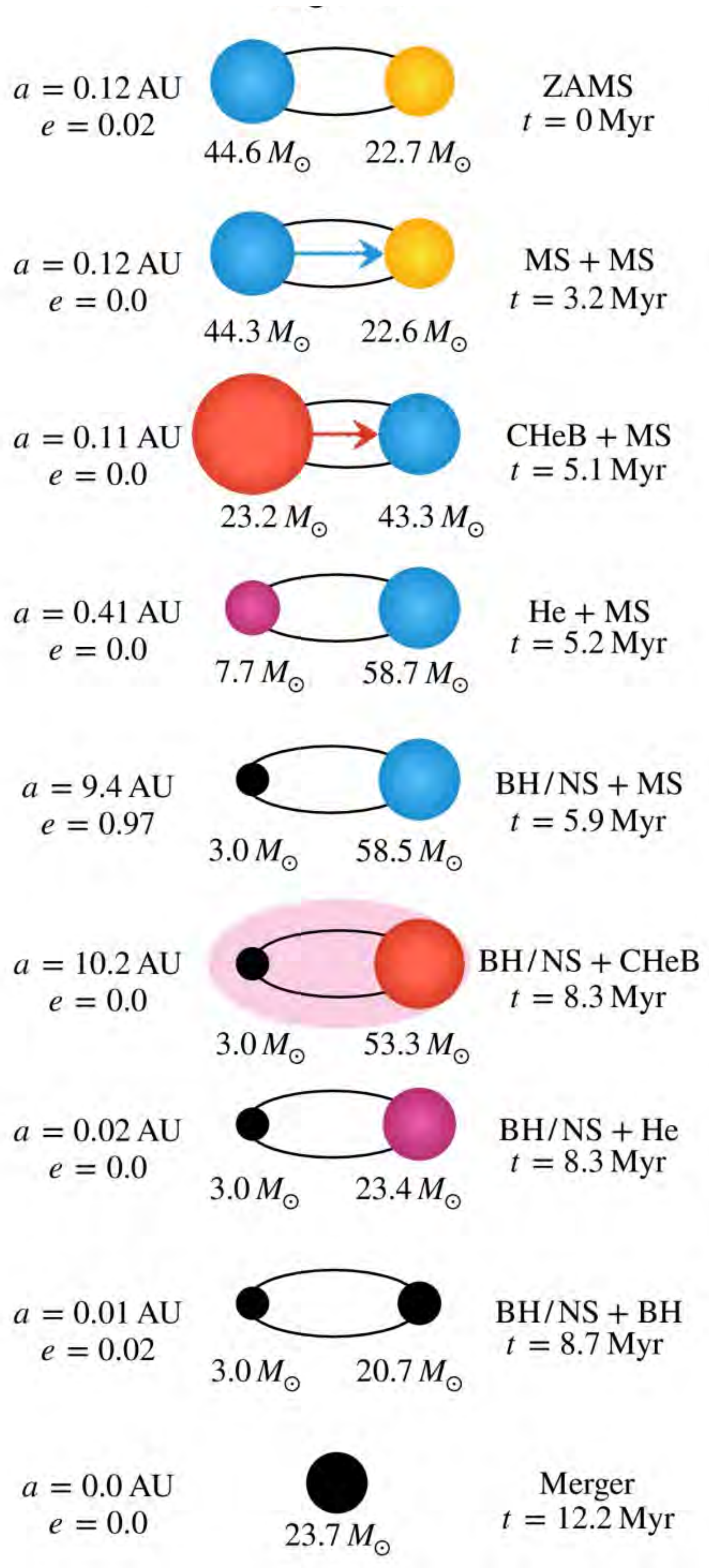
**Eccentric?**

# Stellar remnant binaries - formation channels for NSNS & NSBH



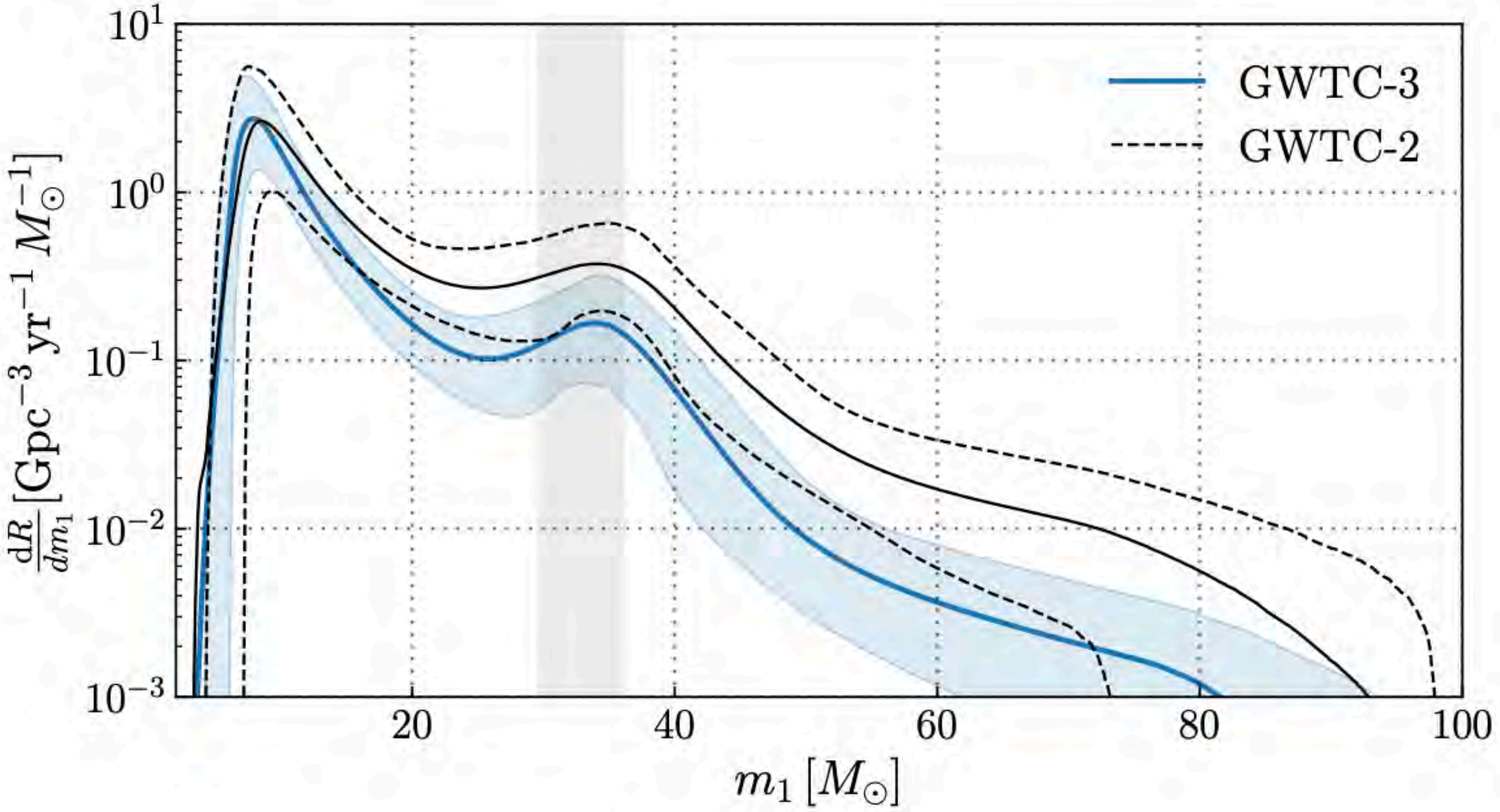

Possible NSNS channel




Possible NSBH channels

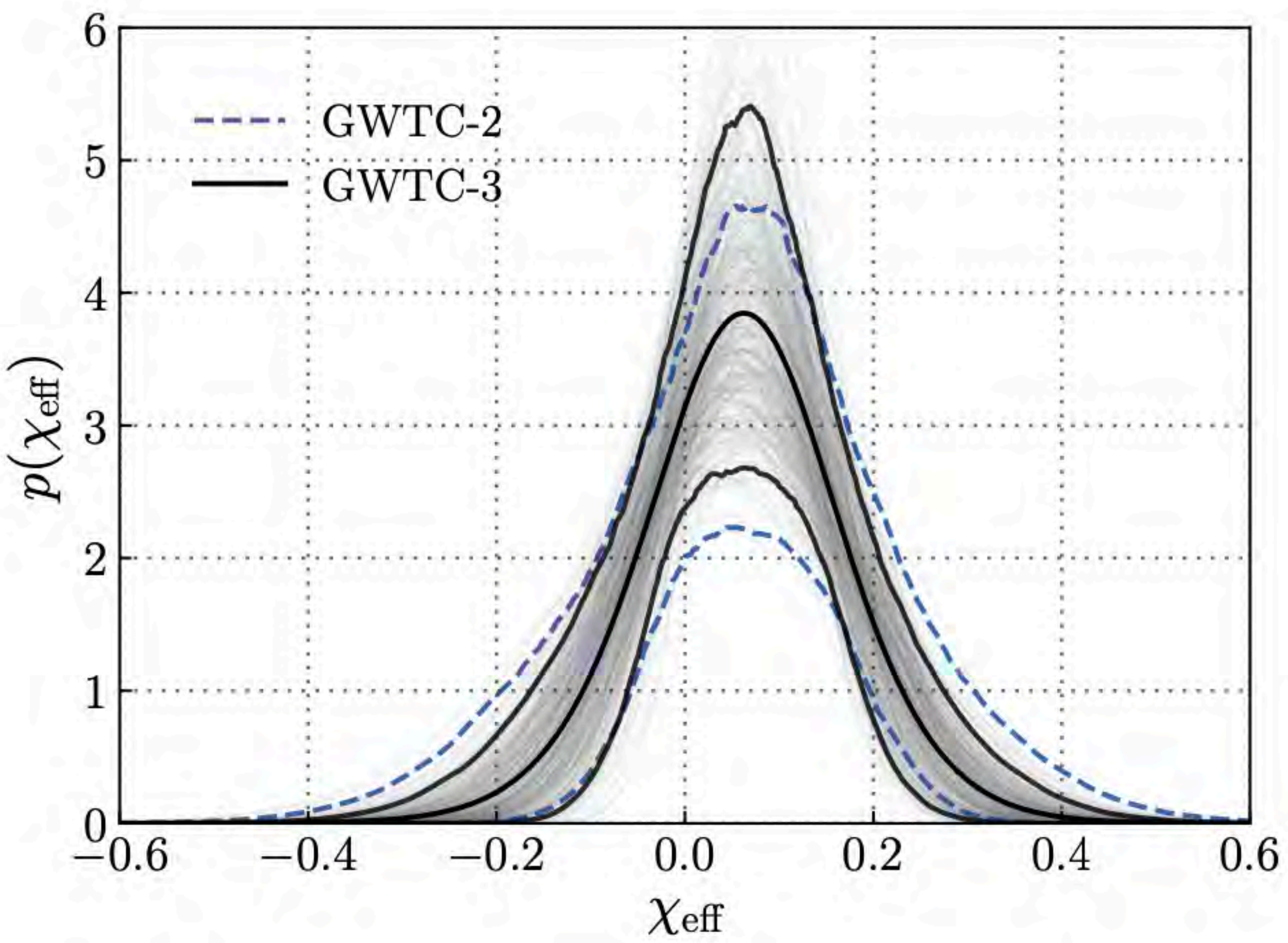
# Resolving features in the BH mass distribution




[LIGO/Virgo 2111.03634]

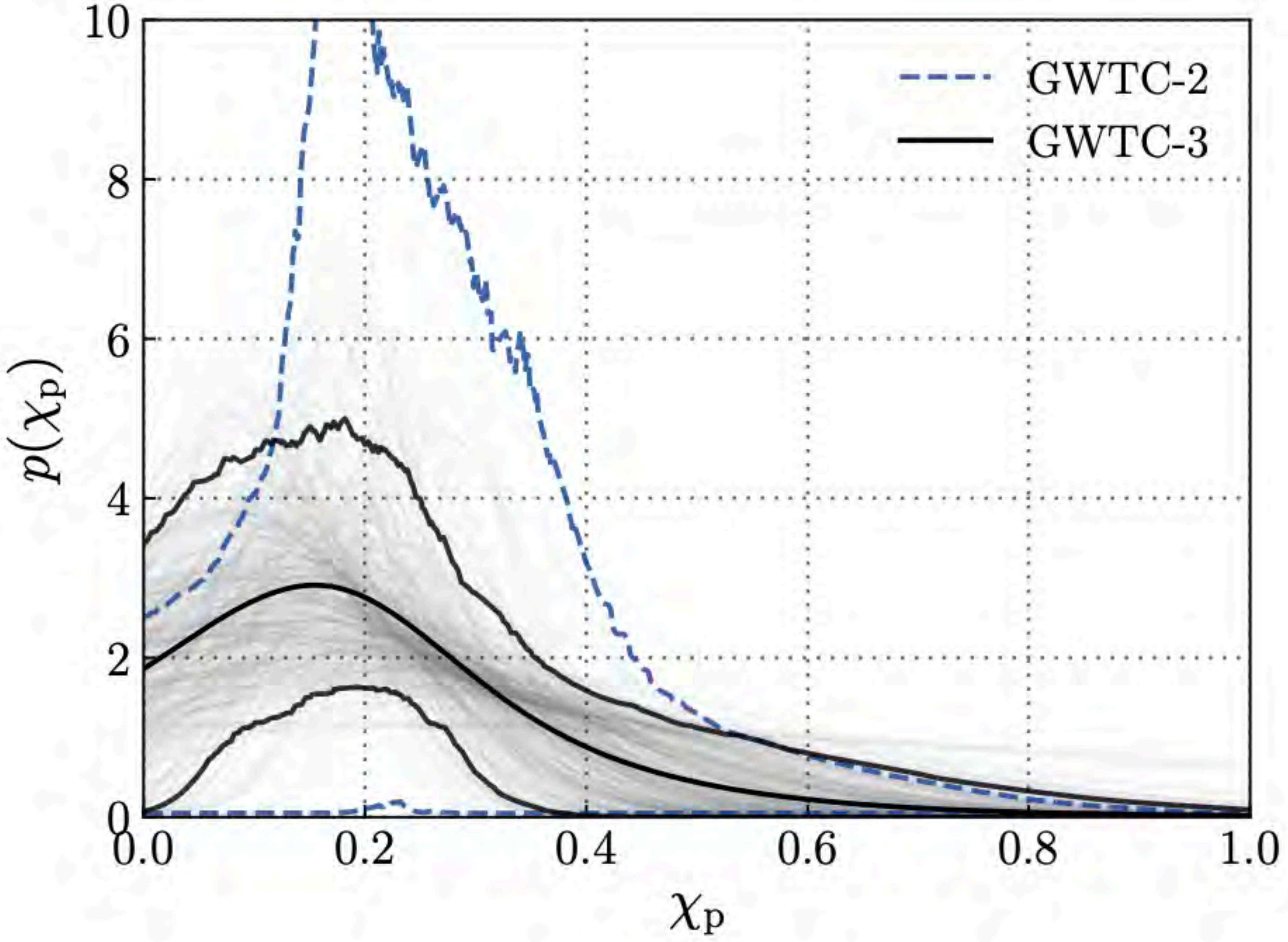
# Spin distribution

Out of orbital plane spin




$$\chi_{\rm eff} = \frac{m_1(\boldsymbol{\chi}_1 \cdot \hat{\mathbf{L}}) + m_2(\boldsymbol{\chi}_2 \cdot \hat{\mathbf{L}})}{m_1 + m_2}$$

In orbital plane spin

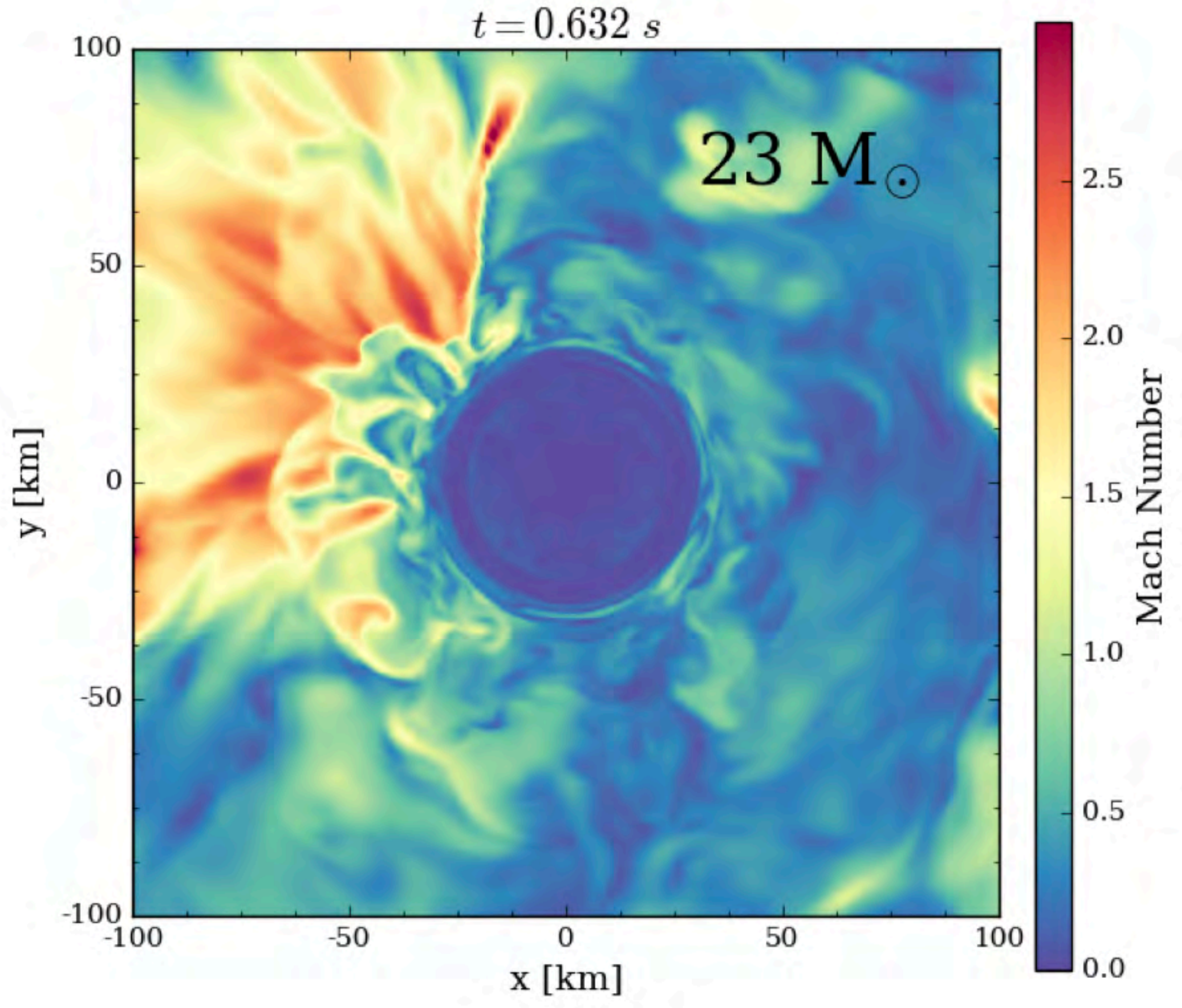


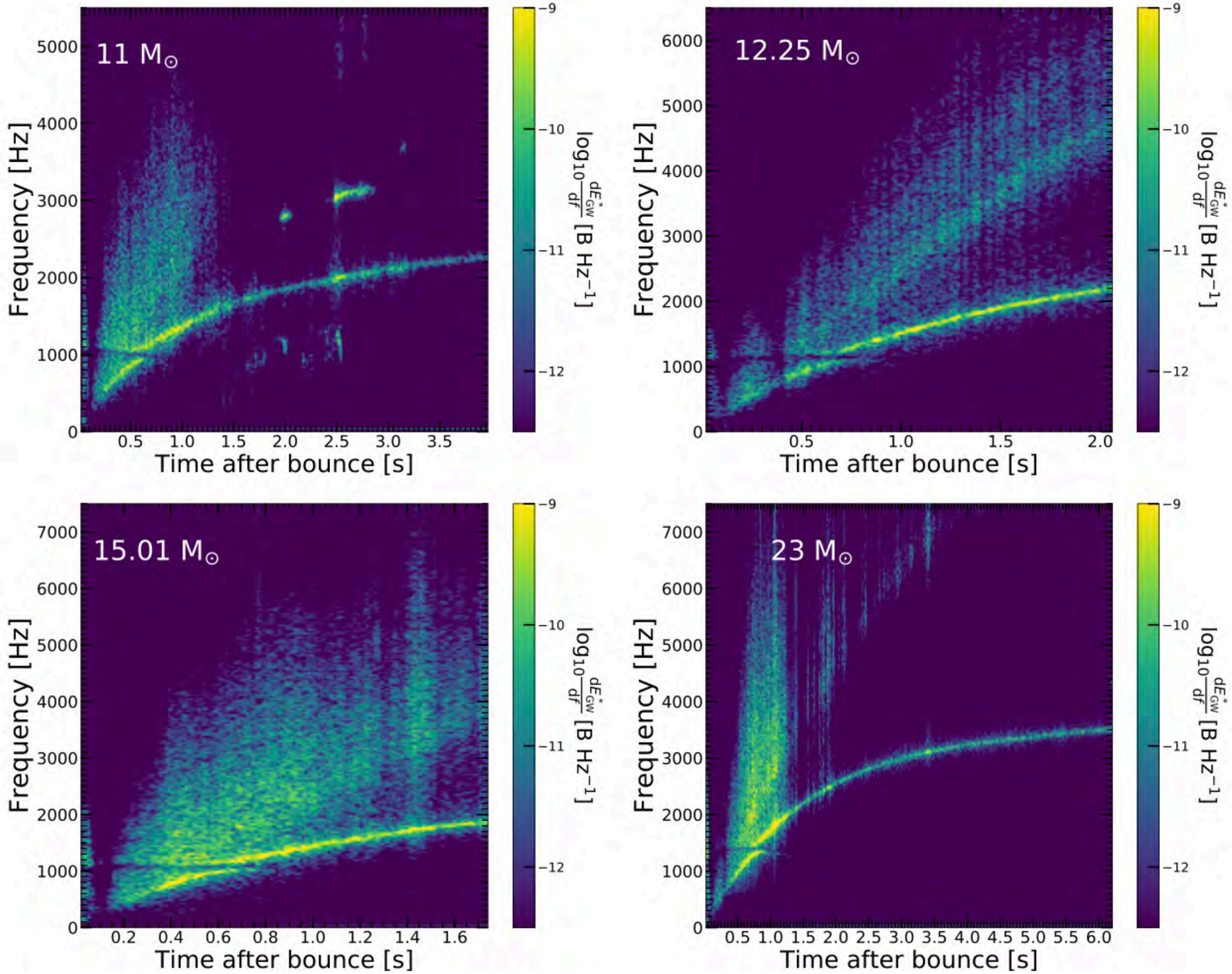

$$\chi_{\rm p} = \frac{1}{2}\left(\chi_{2\perp} + \alpha\,\chi_{1\perp} + |\chi_{2\perp} - \alpha\,\chi_{1\perp}|\right)$$

$$\alpha = \left(\frac{m_1}{m_2}\right)\frac{(4M - m_2)}{(4M - m_1)}$$

[LIGO/Virgo 2111.03634]

# Things we haven't heard yet. Core Collapse SN




[Vartanyan+, ariXiv: 2302.07092]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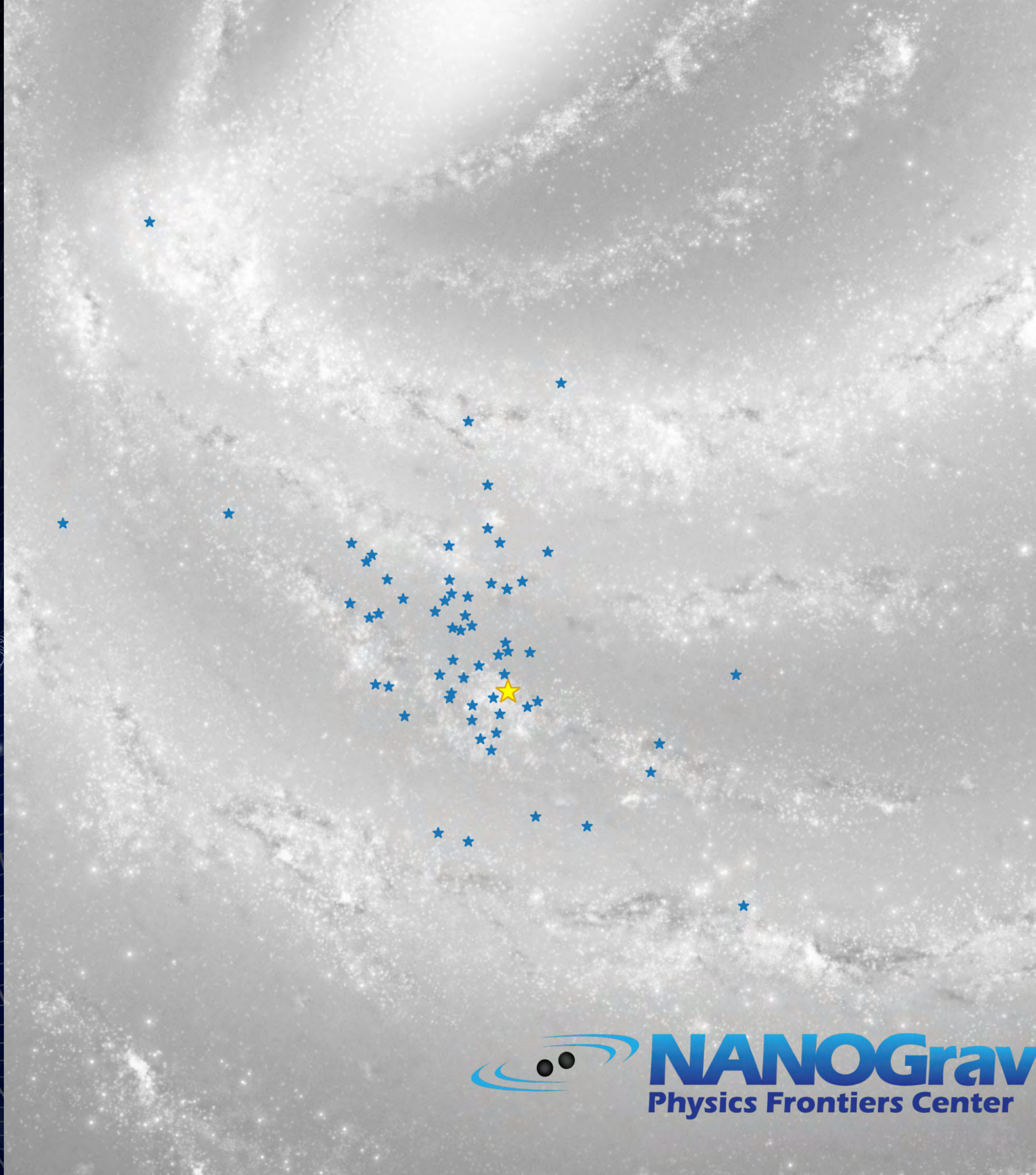
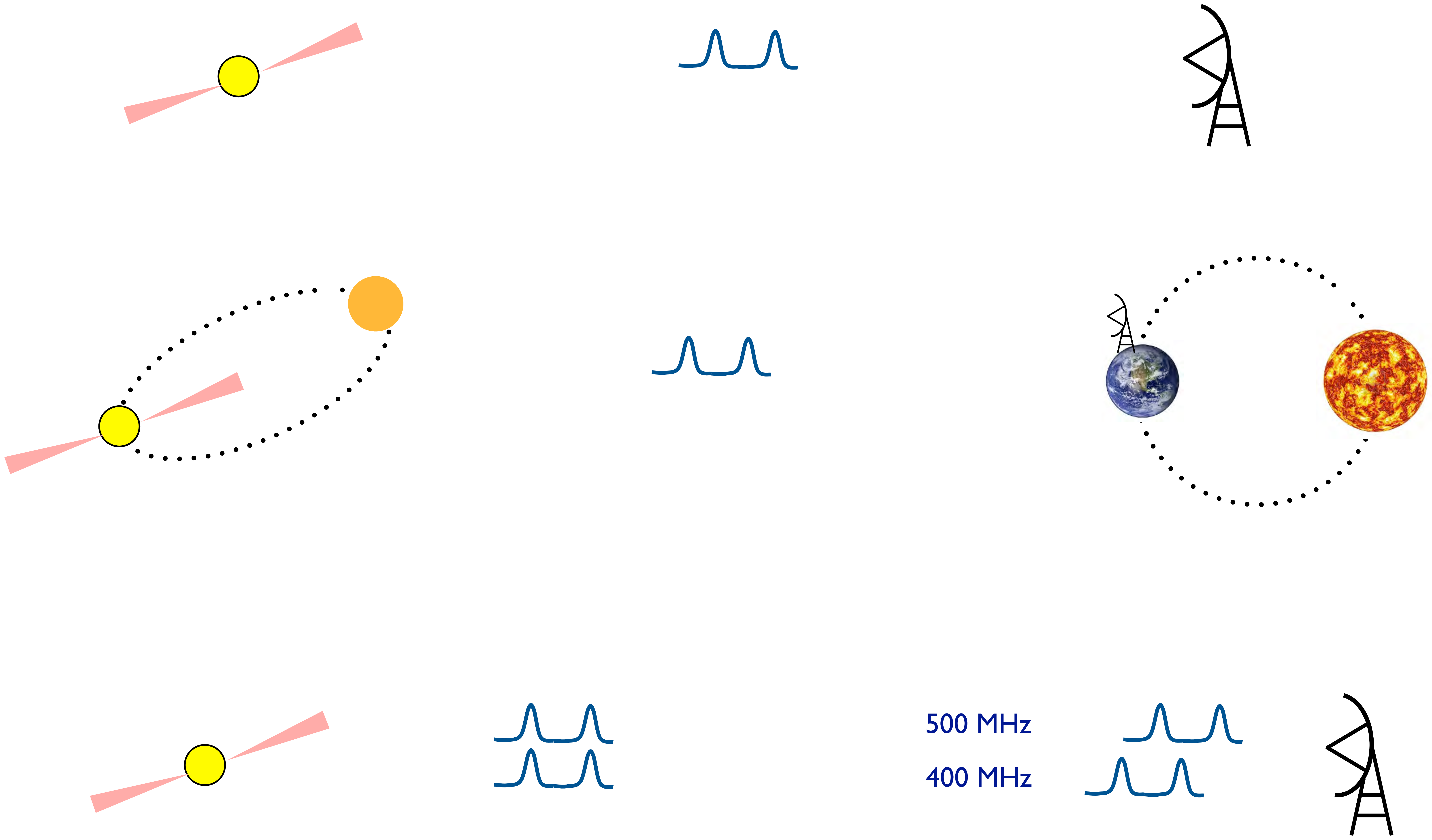
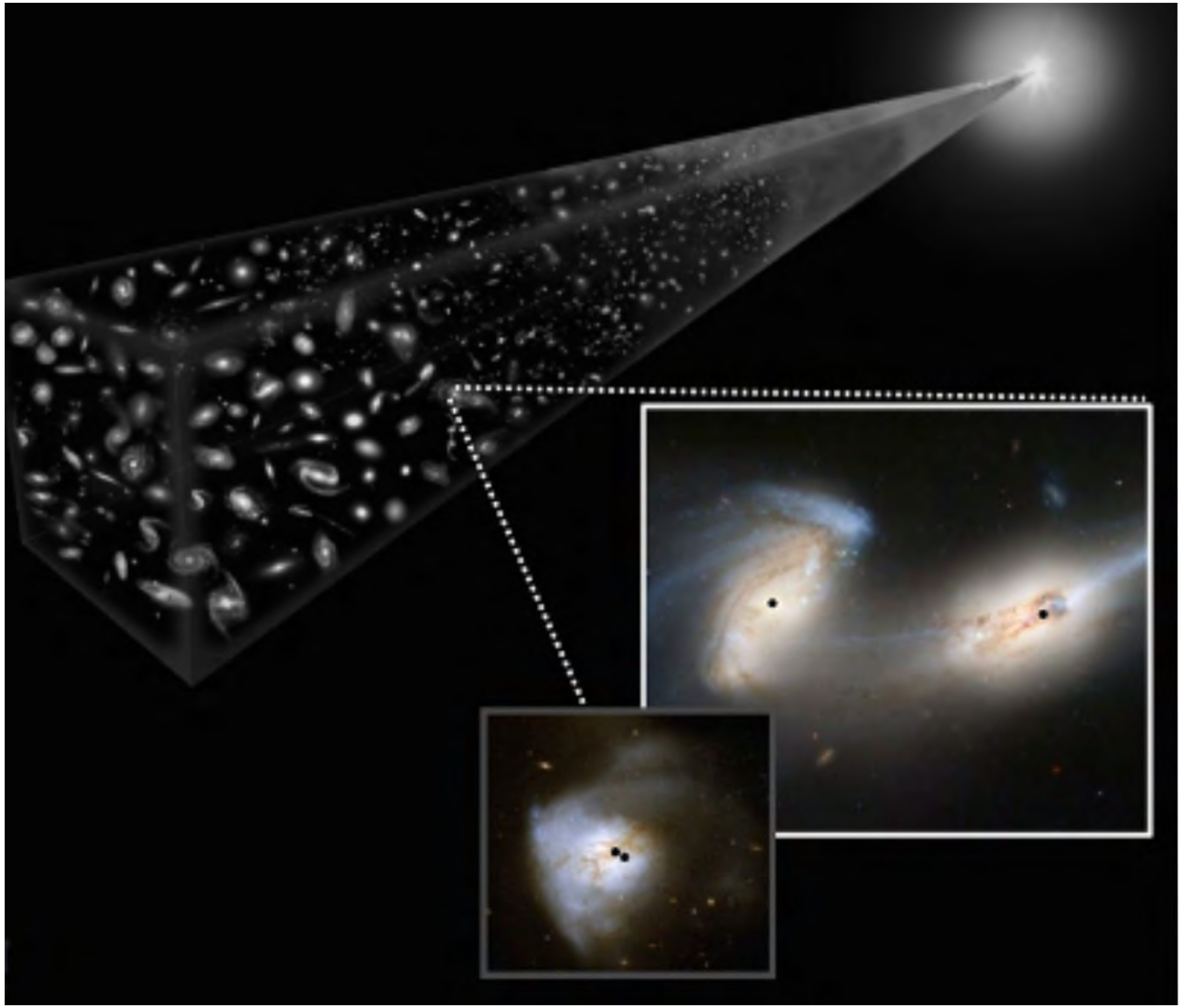

NANOGrav
Physics Frontiers Center

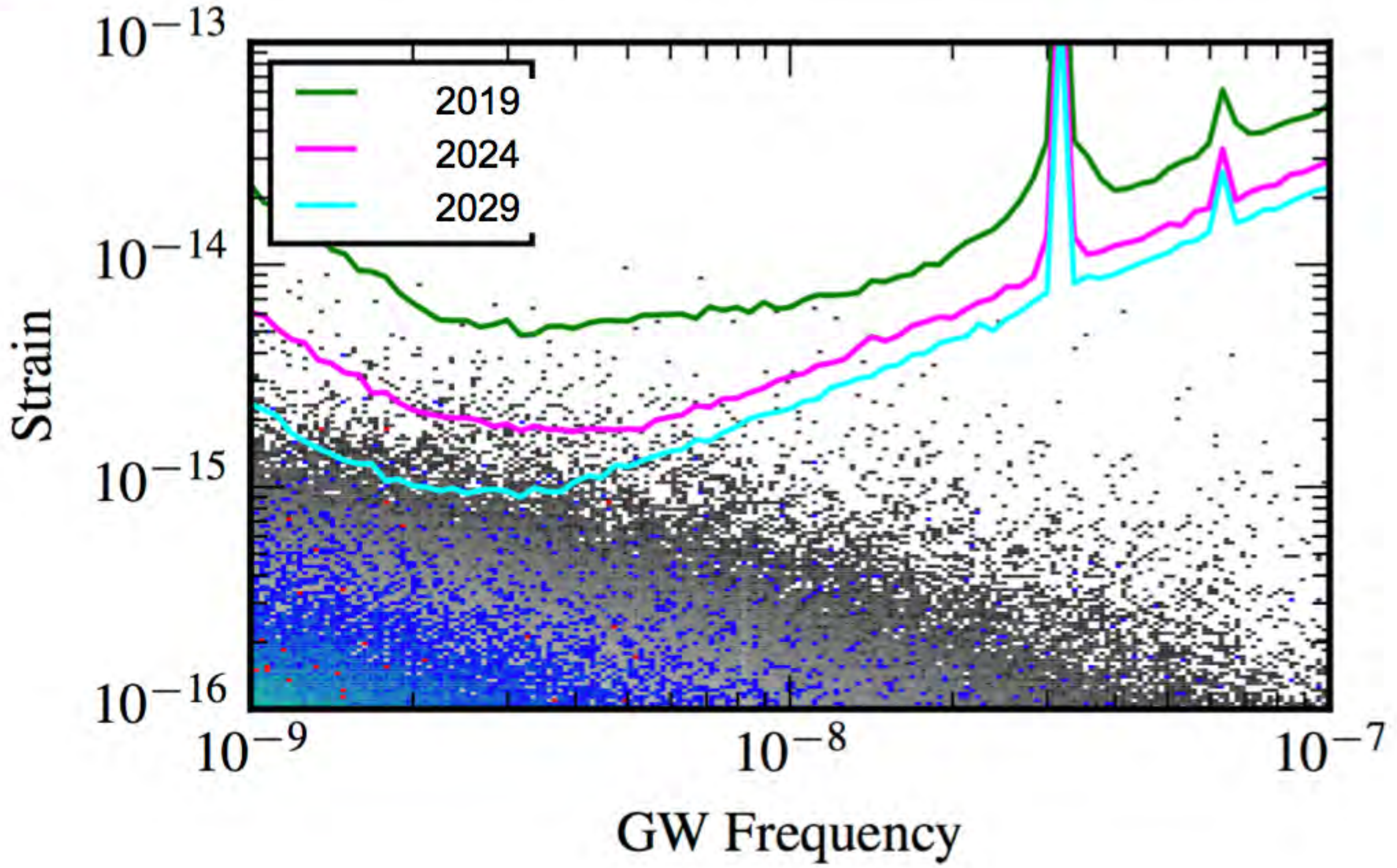
# Pulsar Timing

# Complications....



# Main target: Super massive black hole binaries

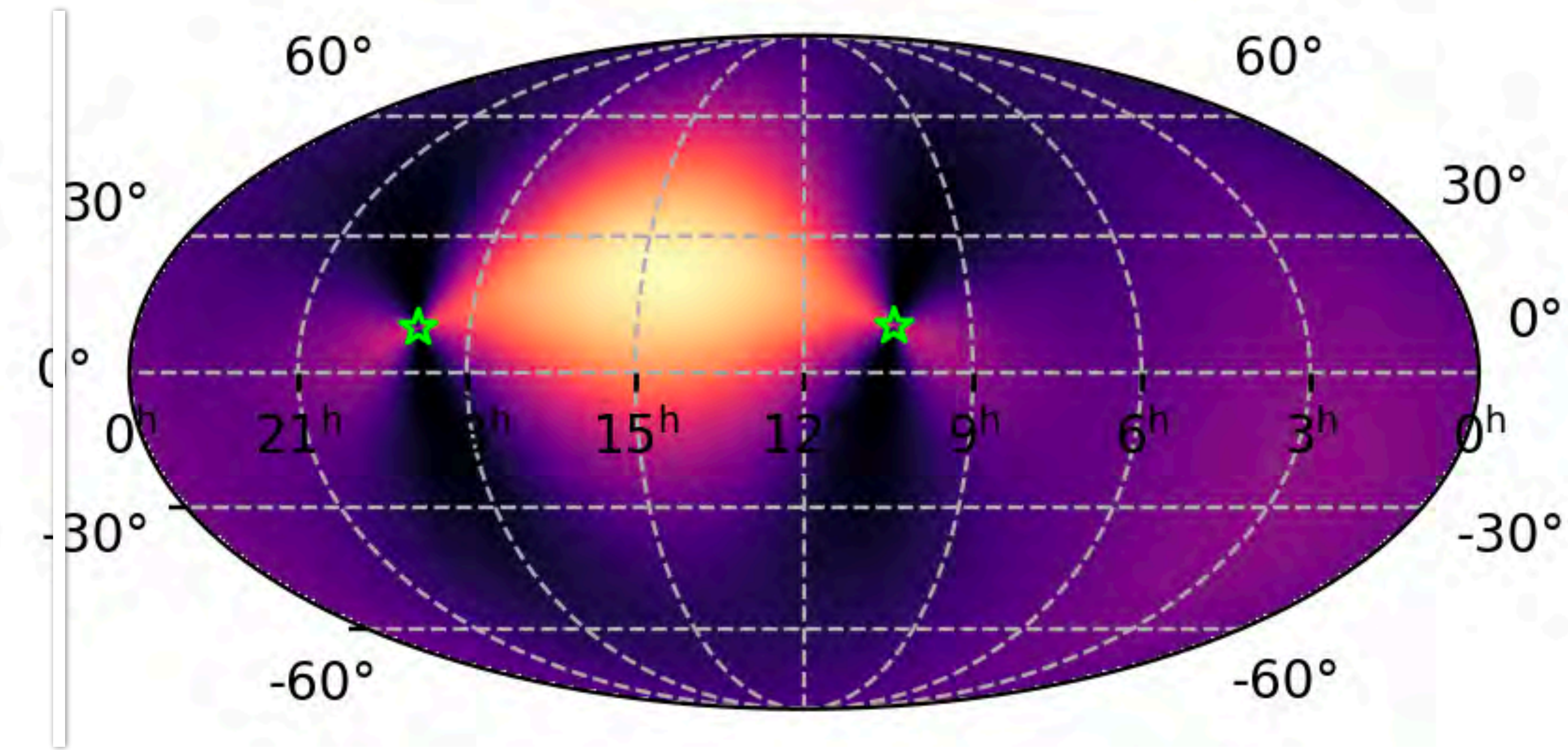




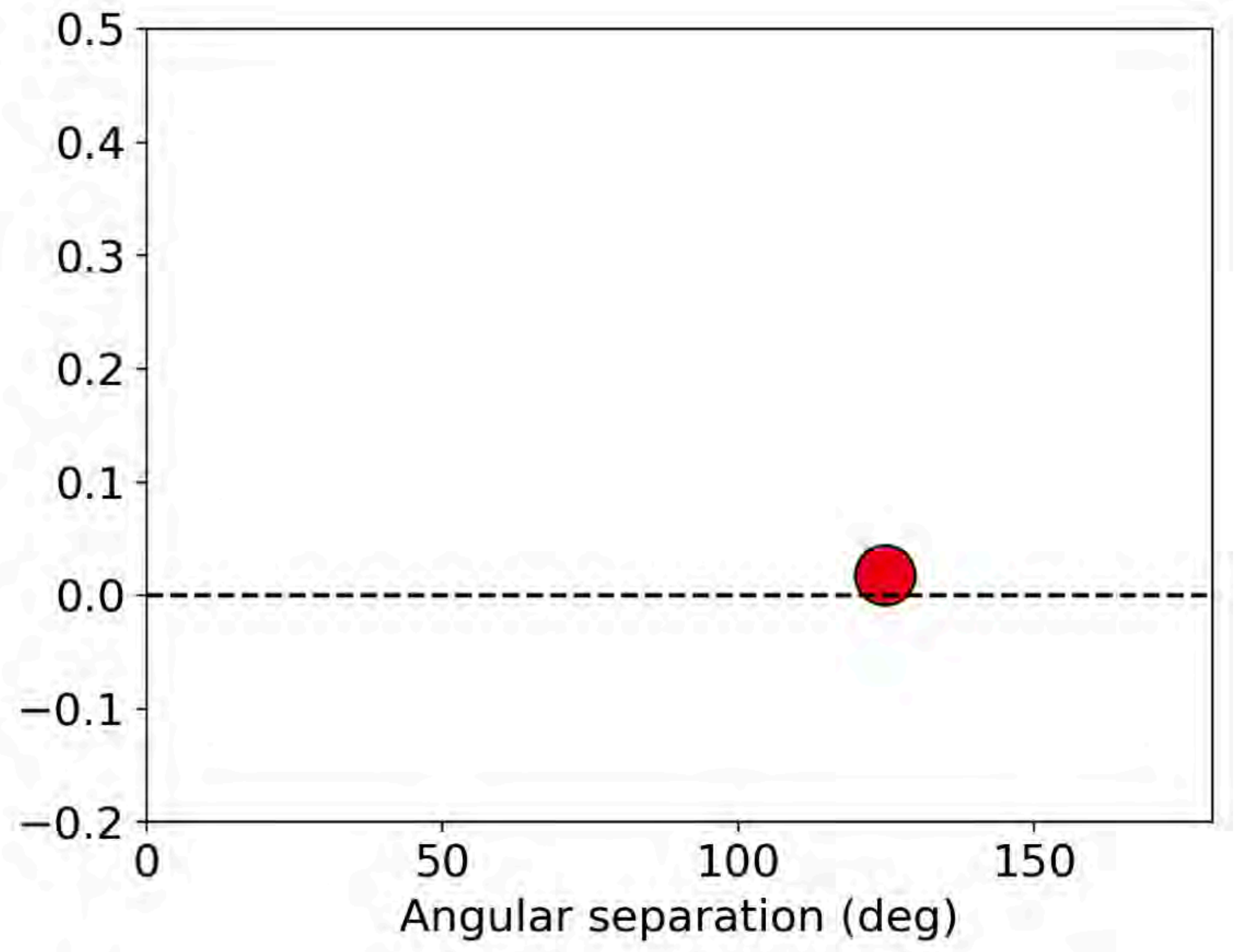

[J. Simon, S. Burke-Spolaor]

# Gravitational Wave Induced Correlations Between Pulsars

correlated responses



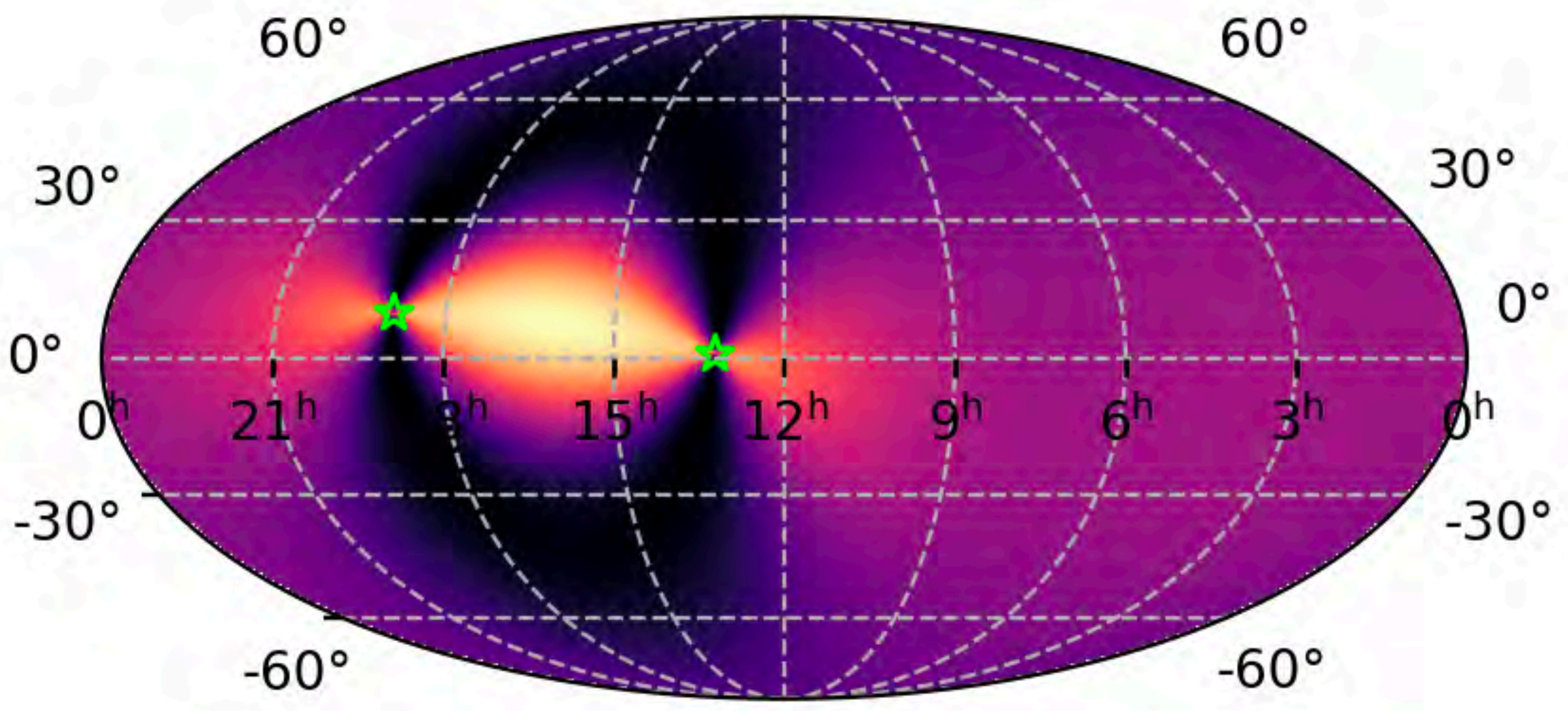

$\int$ power x overlap d(sky)

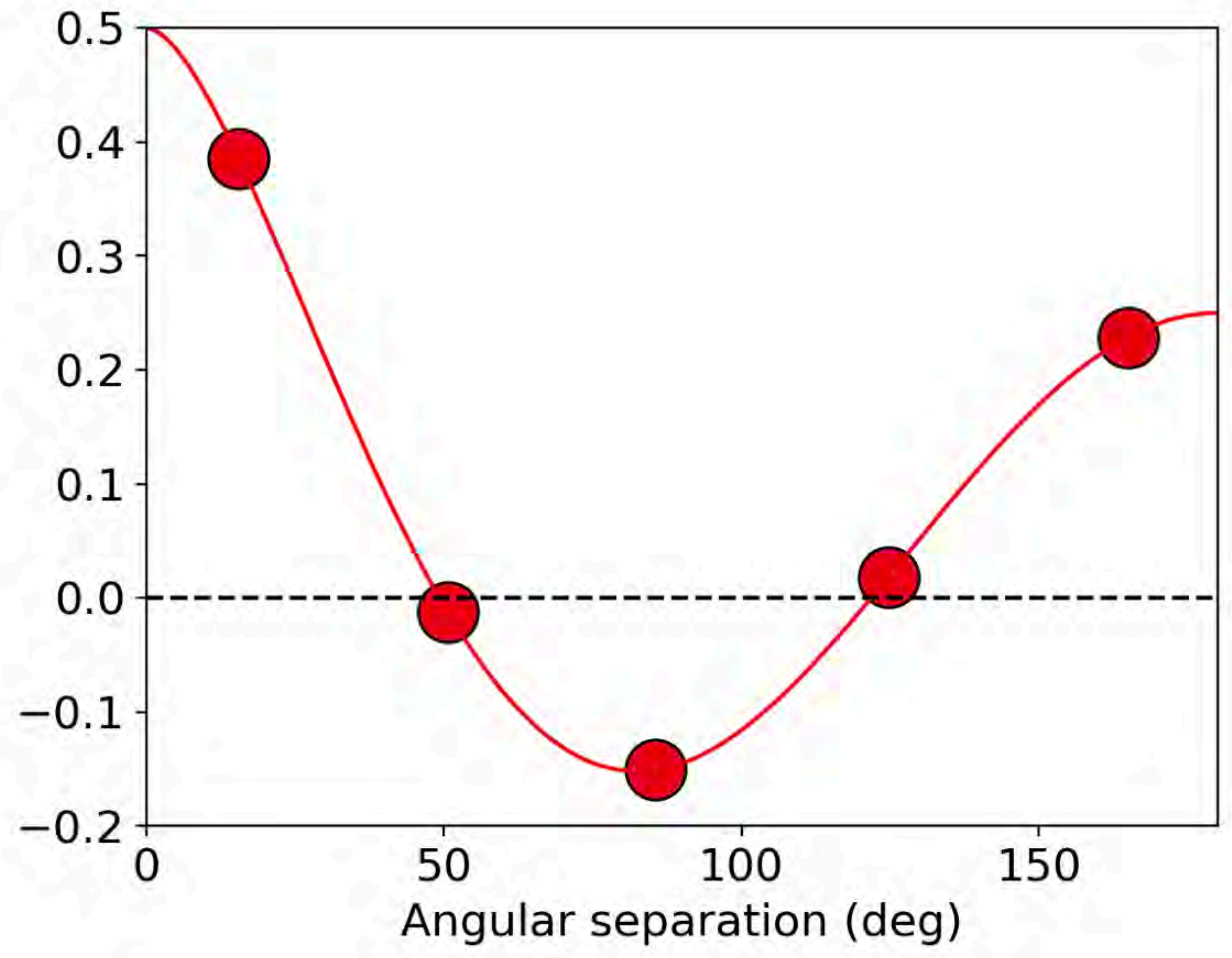



Slide by Steven Taylor

# Gravitational Wave Induced Correlations Between Pulsars

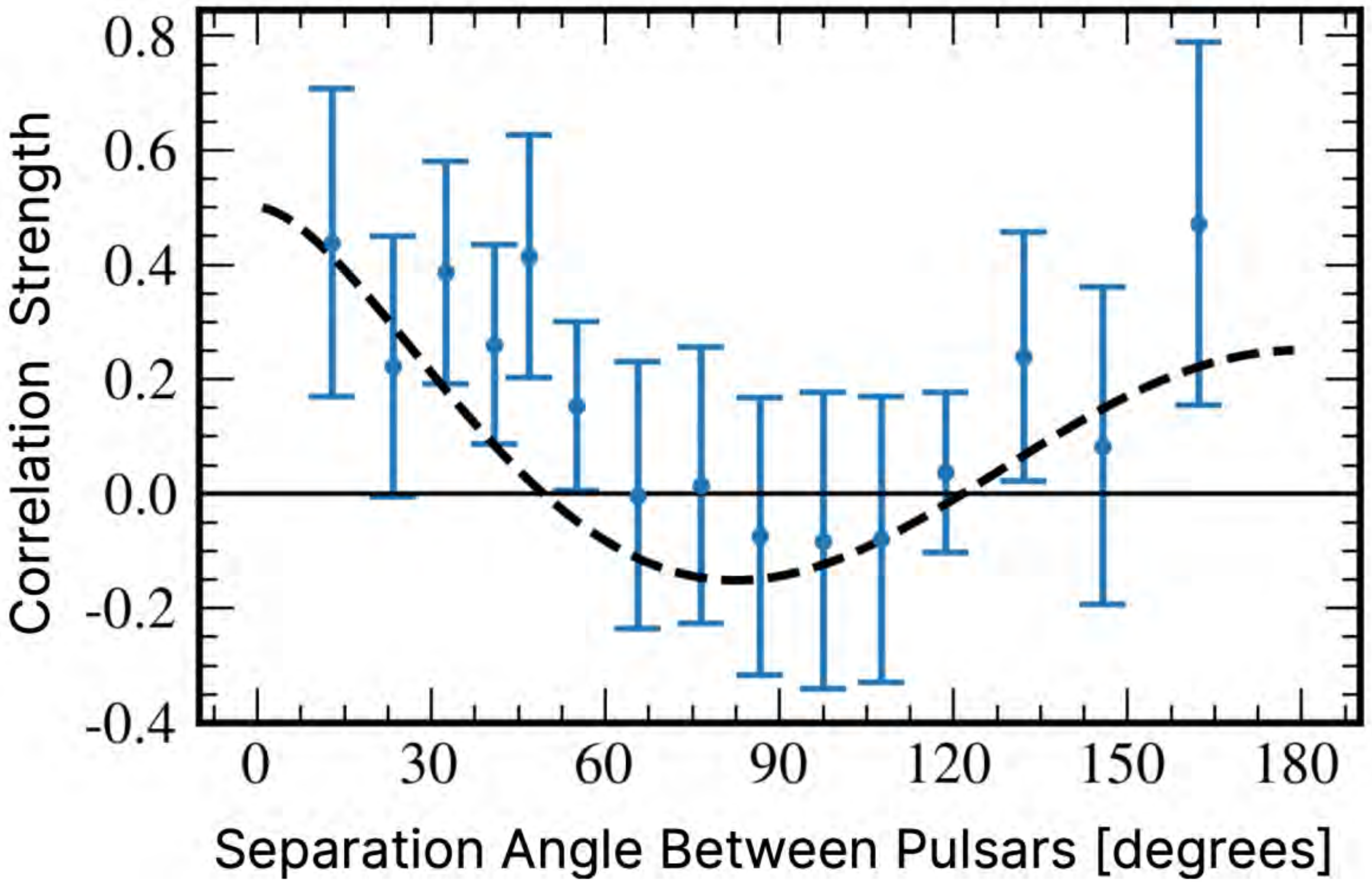
correlated responses



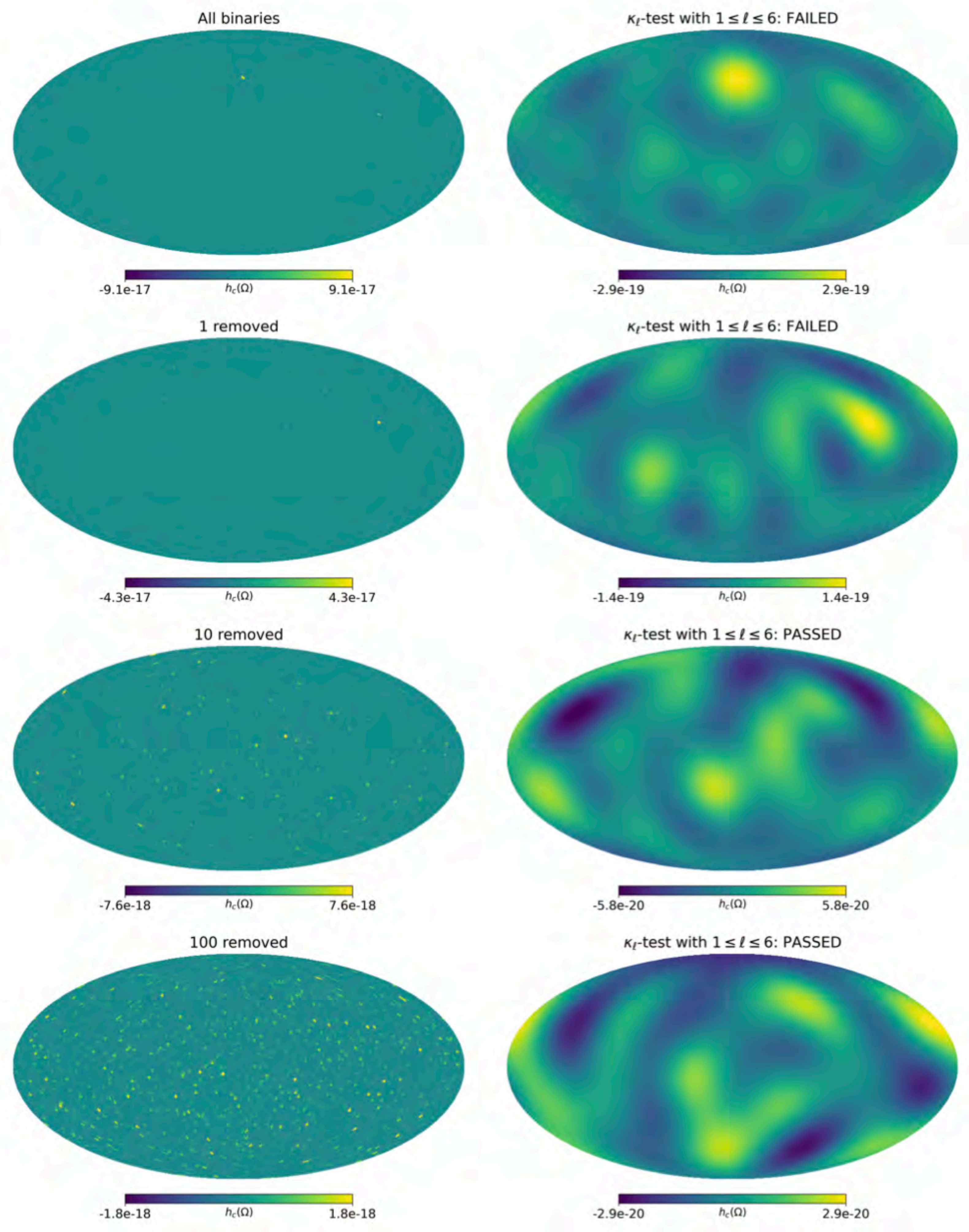




Slide by Steven Taylor

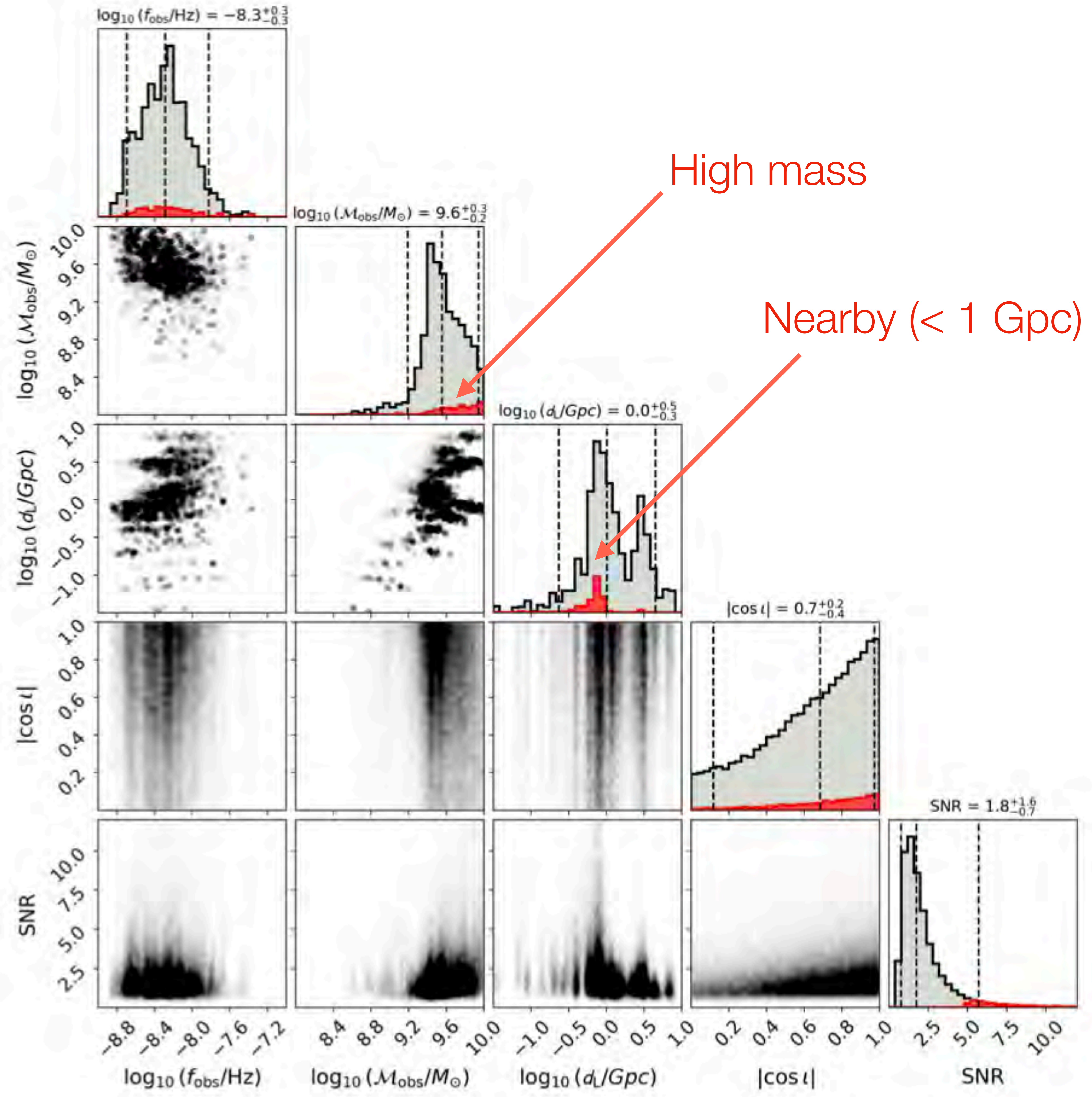
# The NANOGrav 15 yr Data Set: Evidence for a Gravitational-wave Background



# So what is causing the signal?



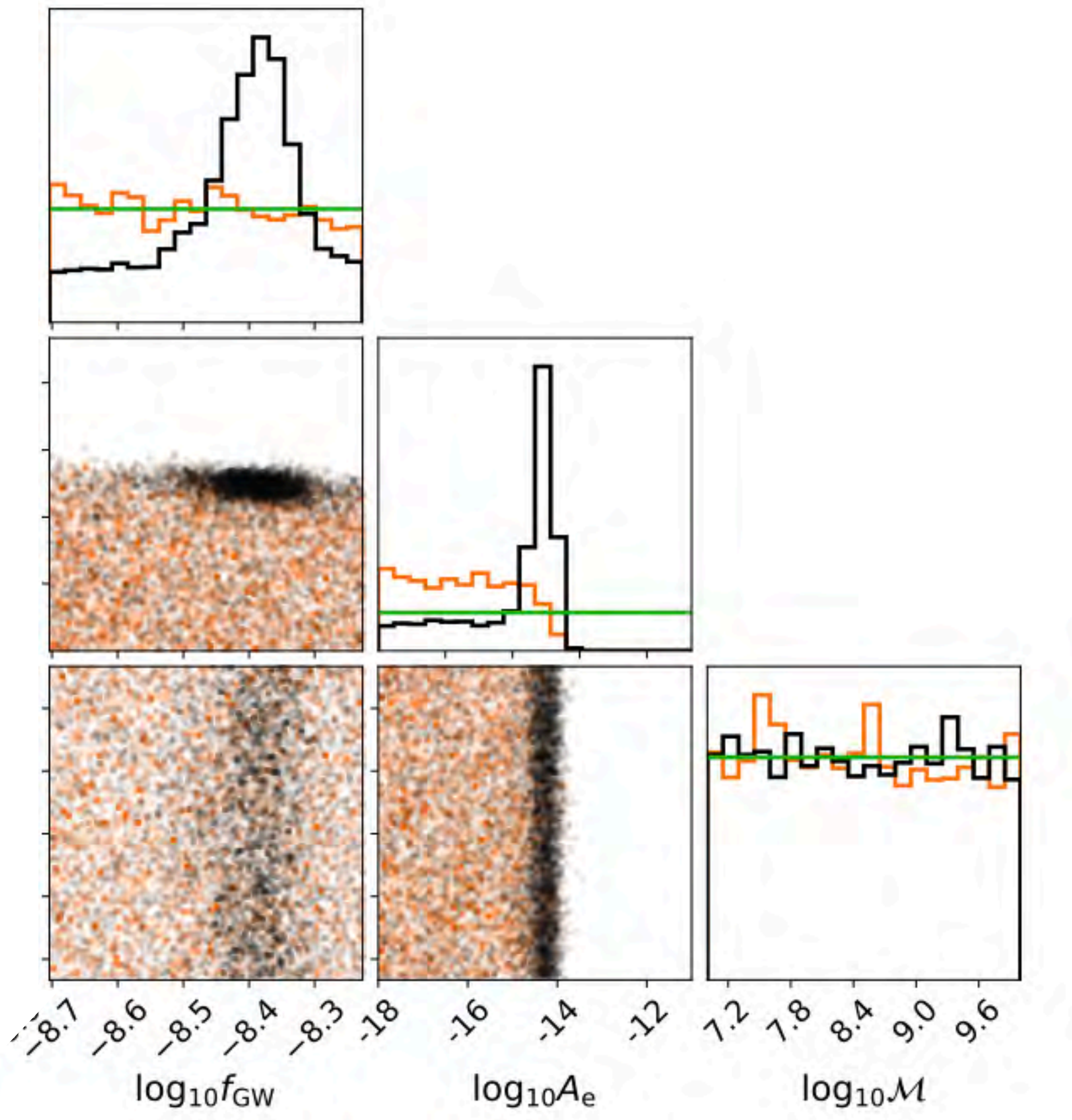

Anisotropy

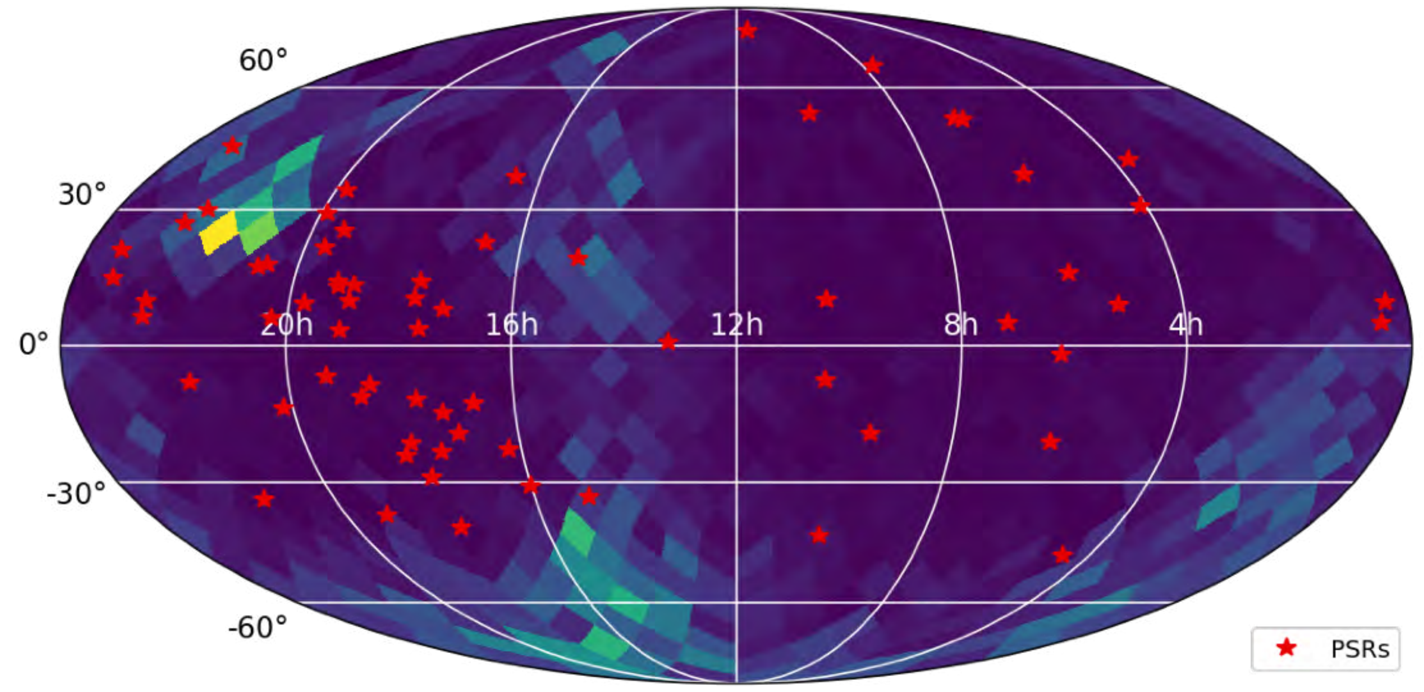



Individual loud sources

# Hint of a black hole binary signal at 4 nHz

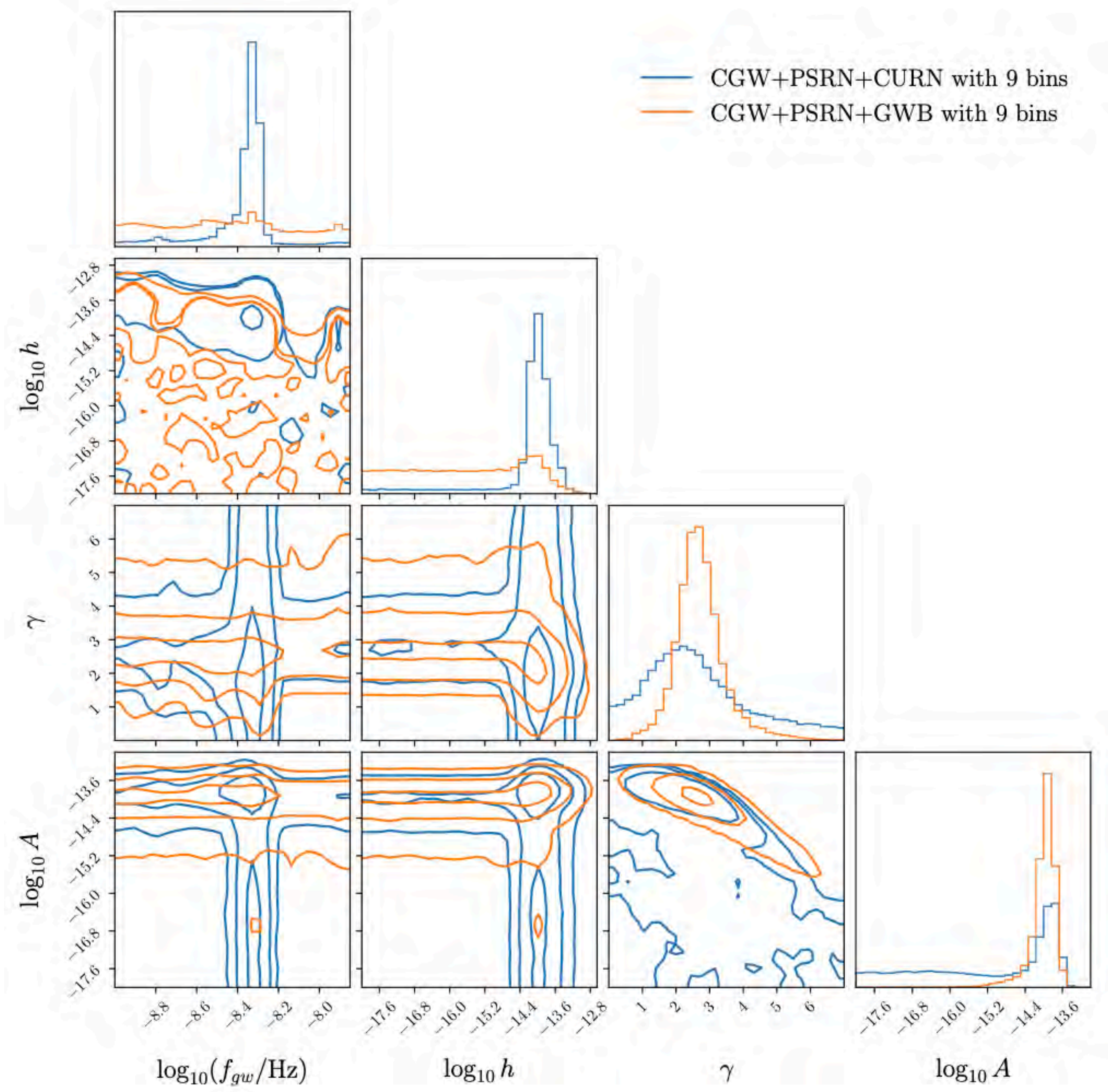
## NANOGrav

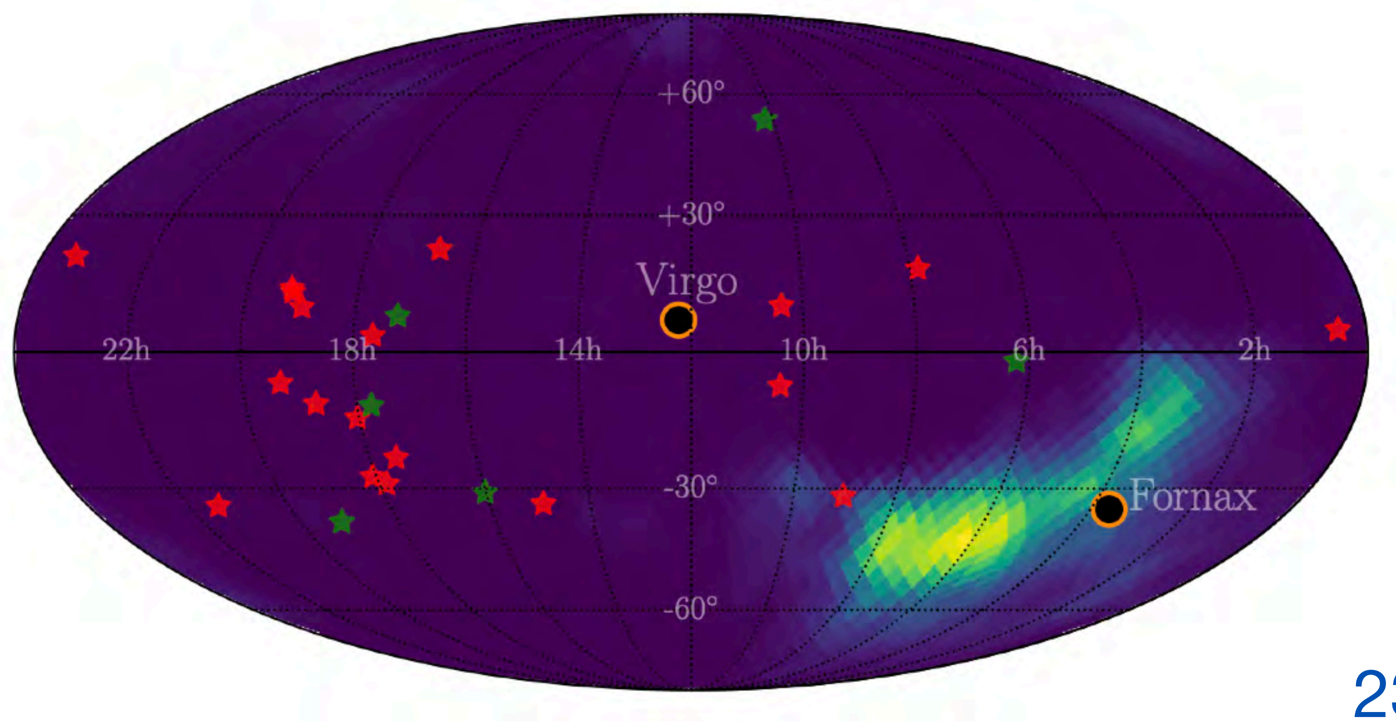



2306.16222

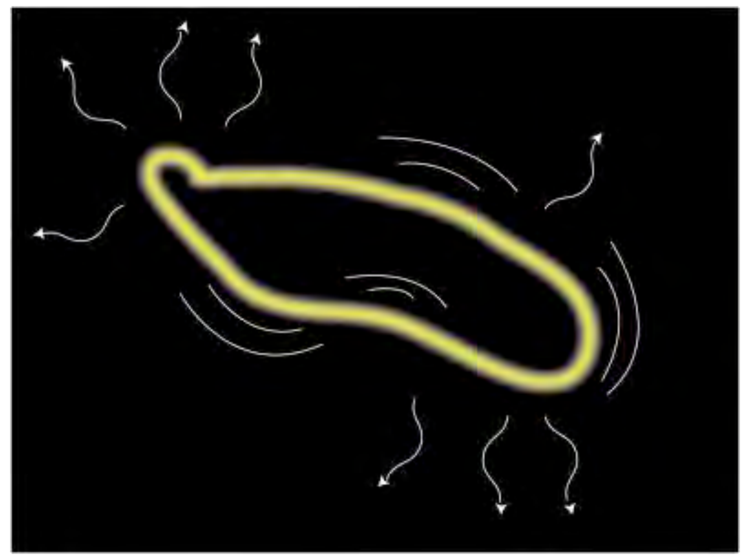
## EPTA

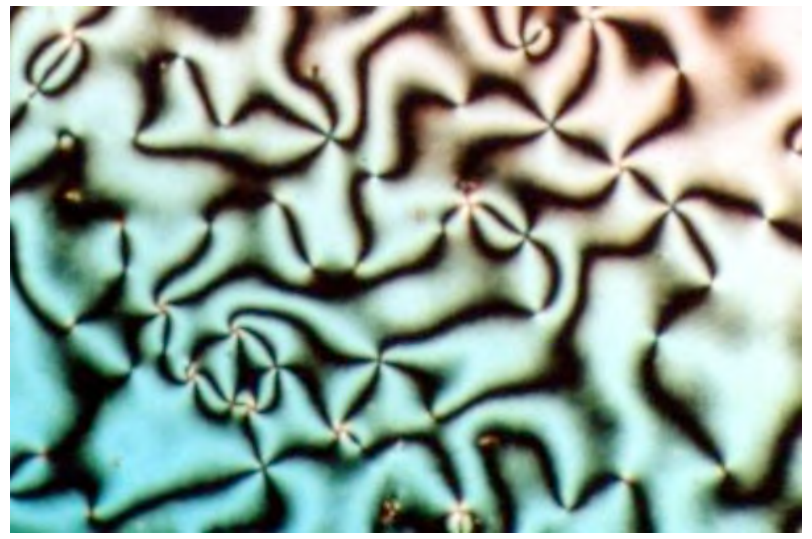


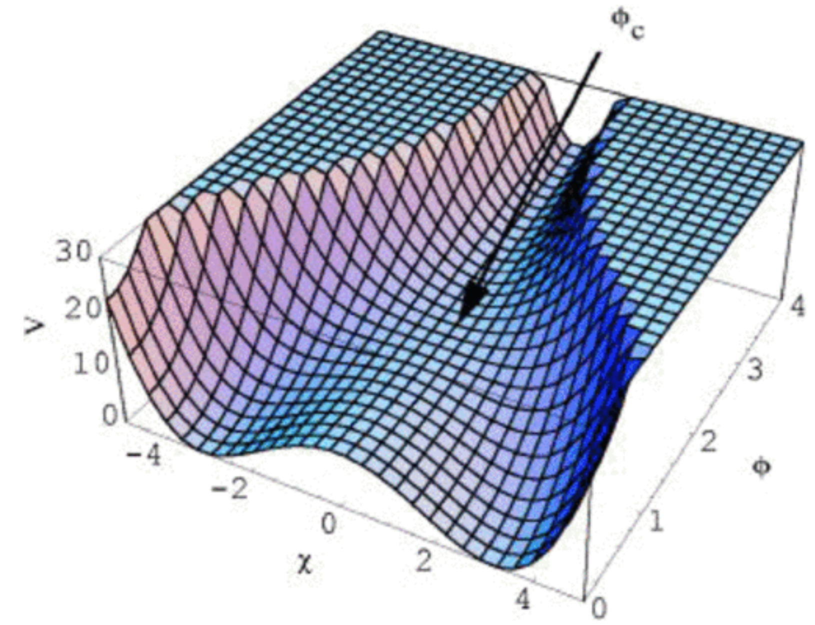

2306.16226

# Exotic Sources





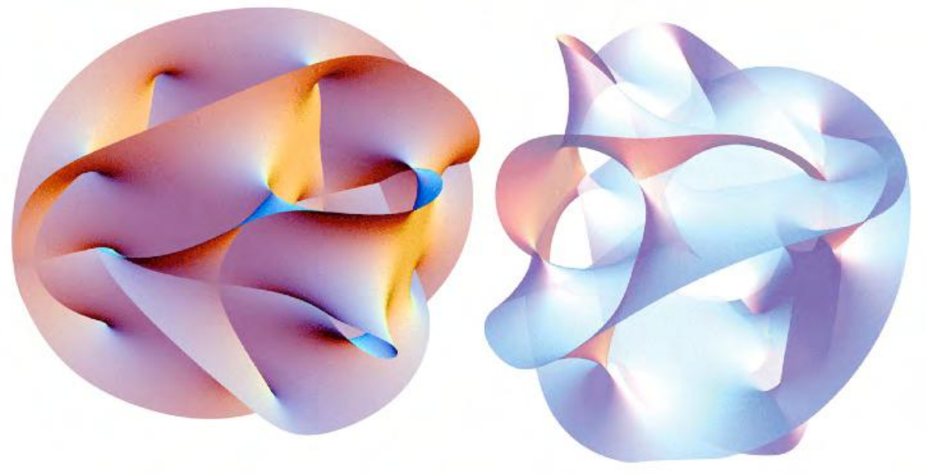
Topological defects



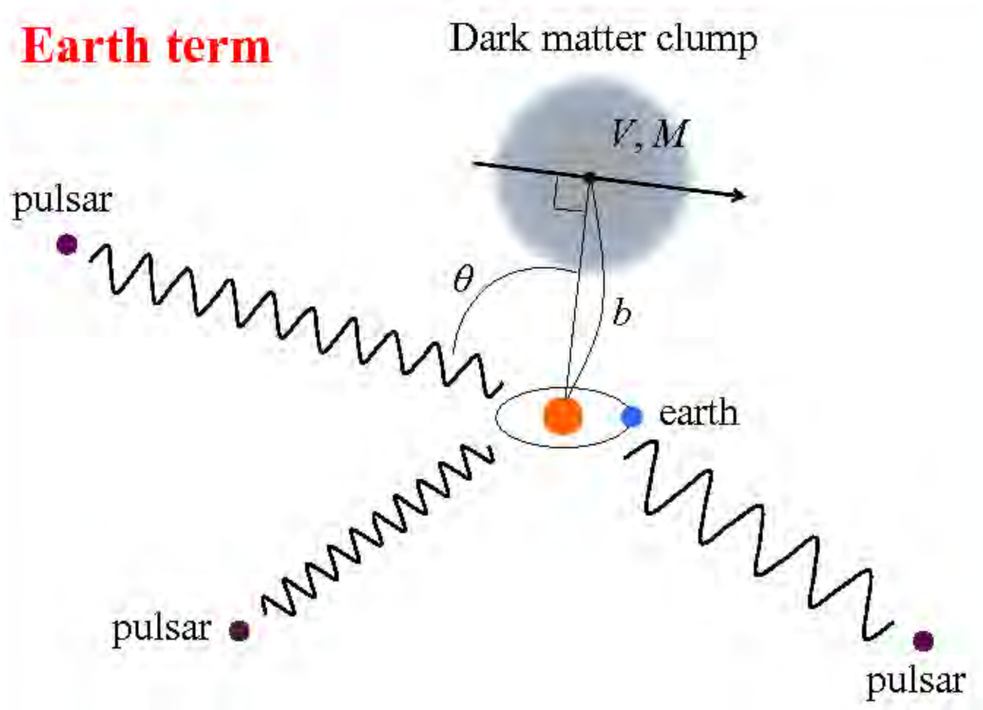

Pre-heating/Re-heating

Warped extra dimensions

Phase transitions- bubble nucleation, cavitation, collisions



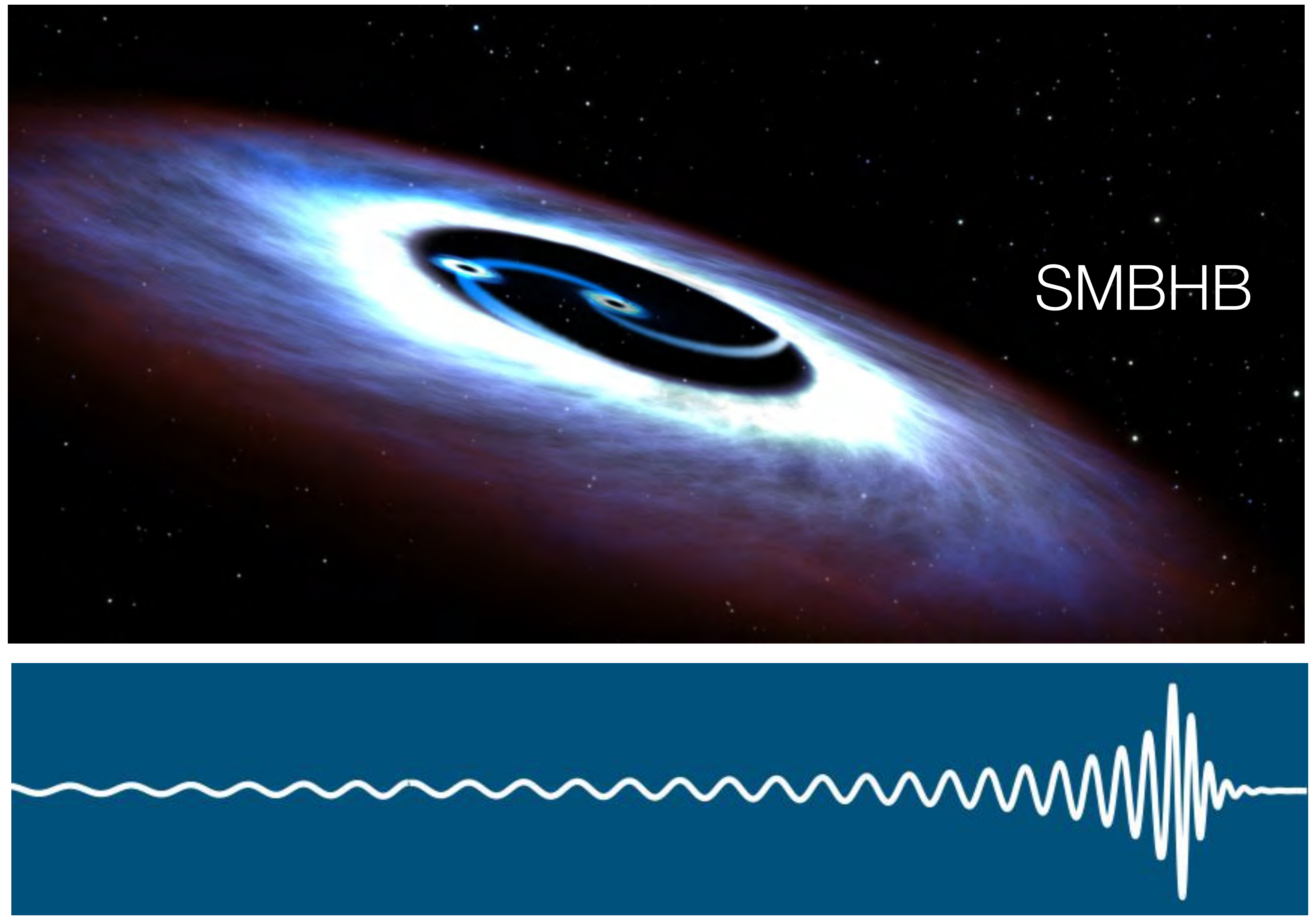
Braneworlds

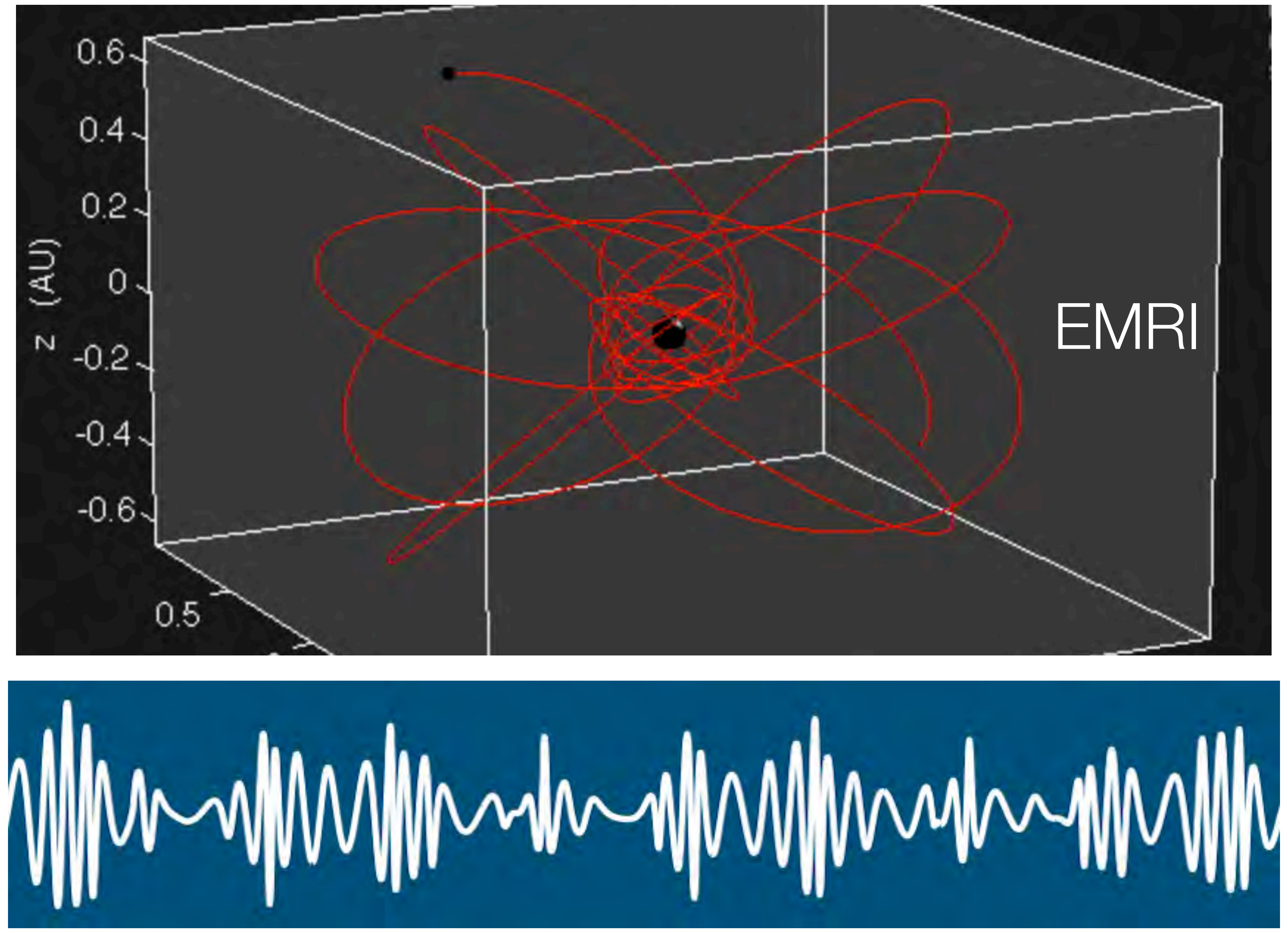


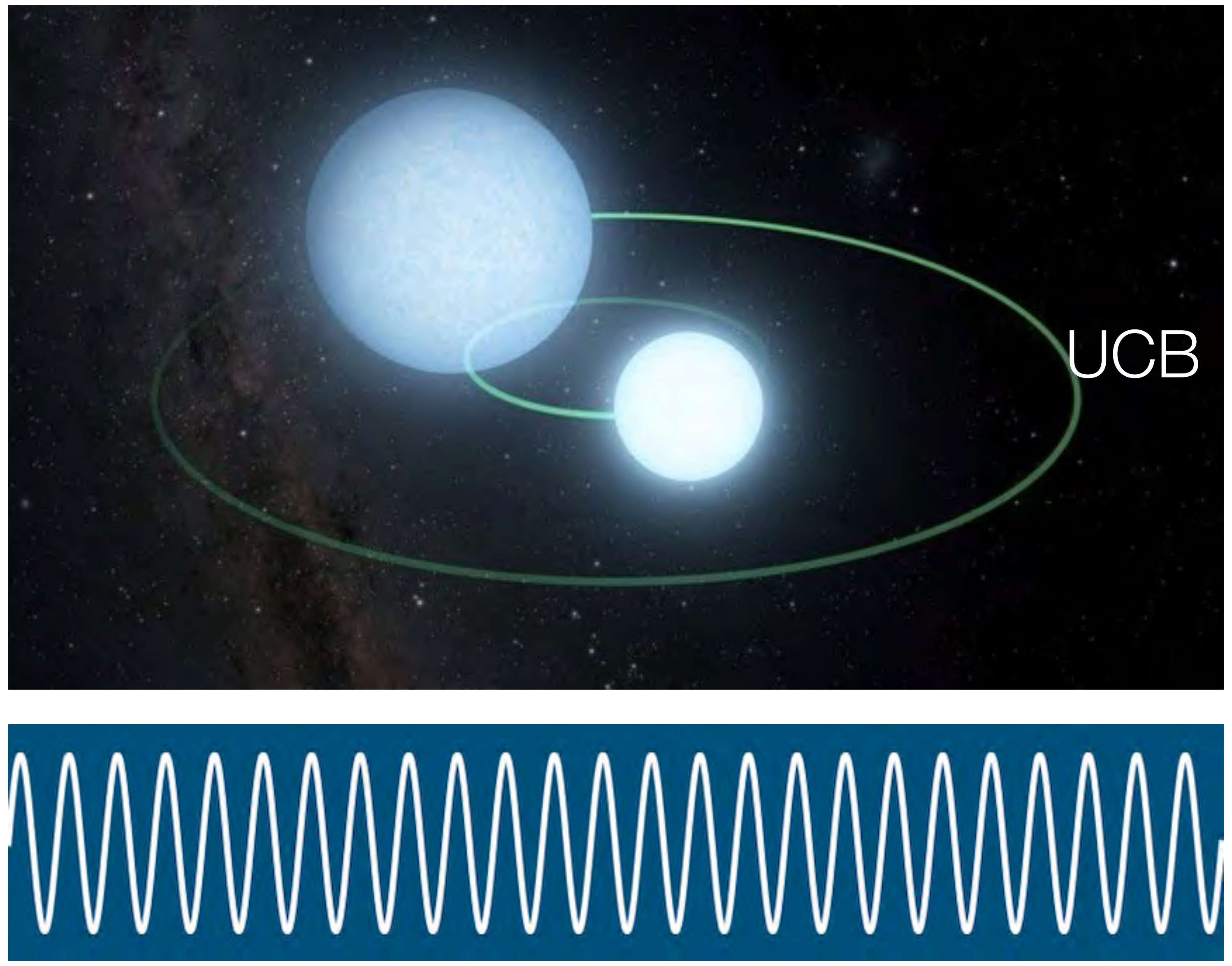

Dark Matter

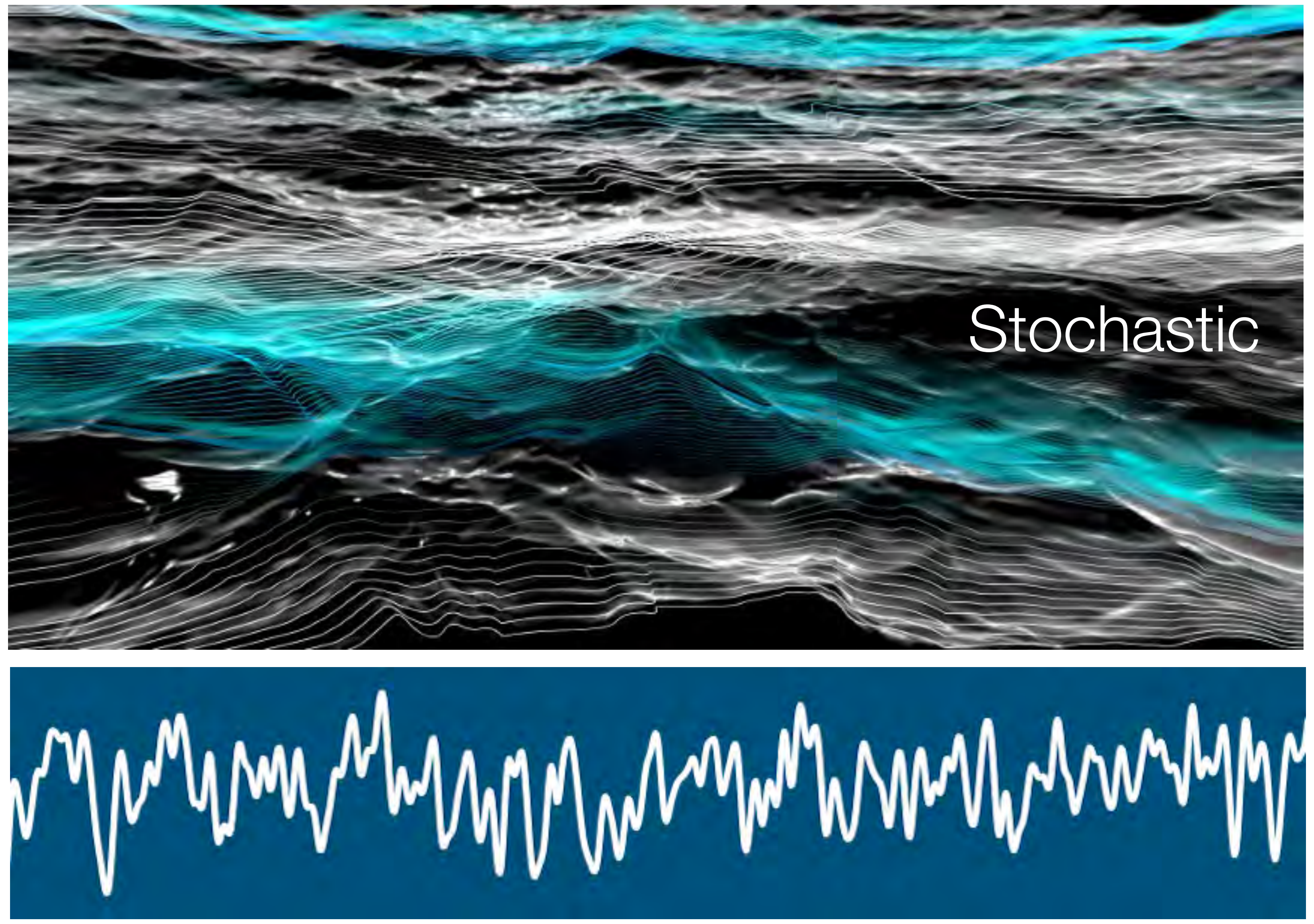
?

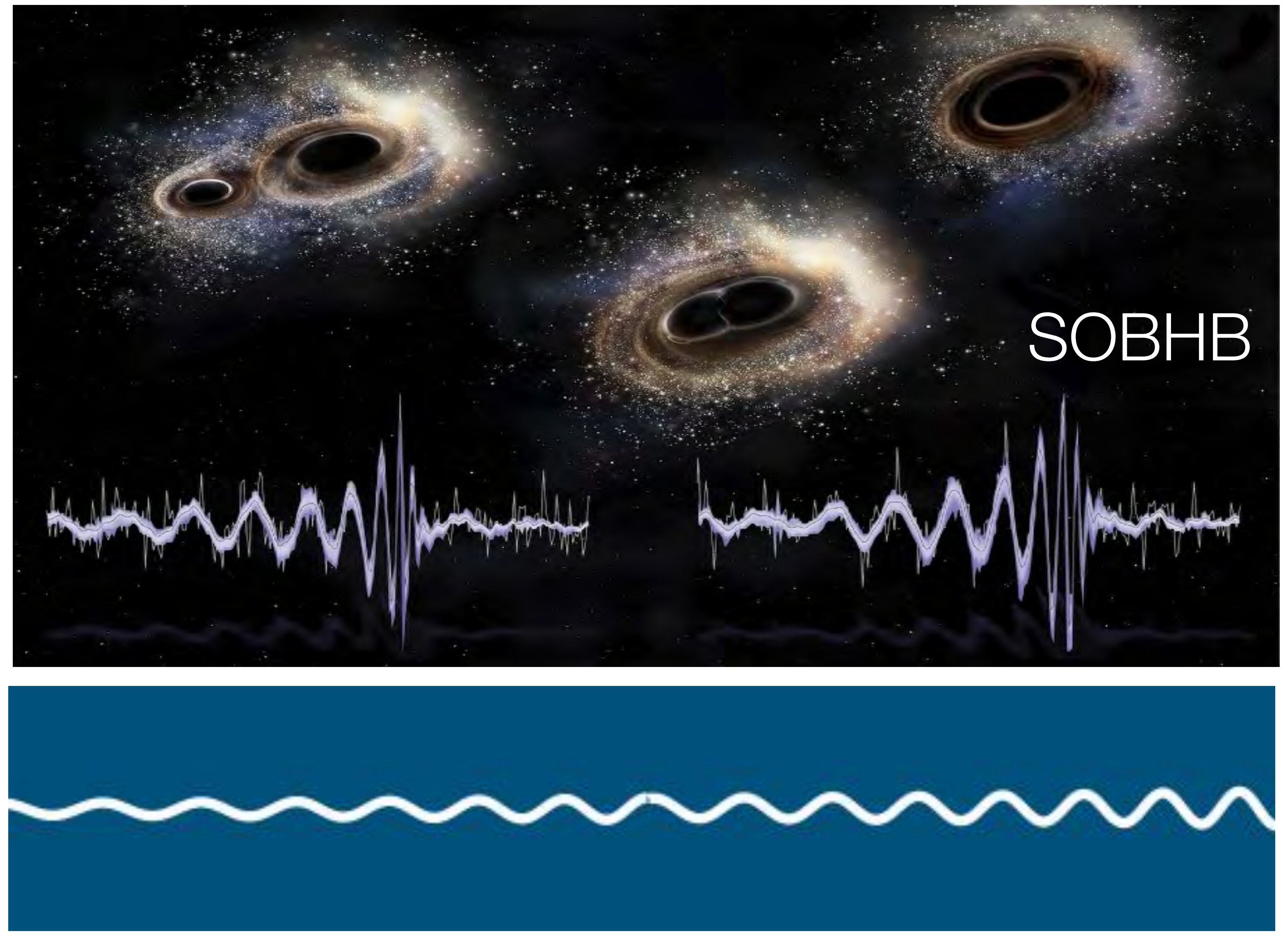
# LISA Sources



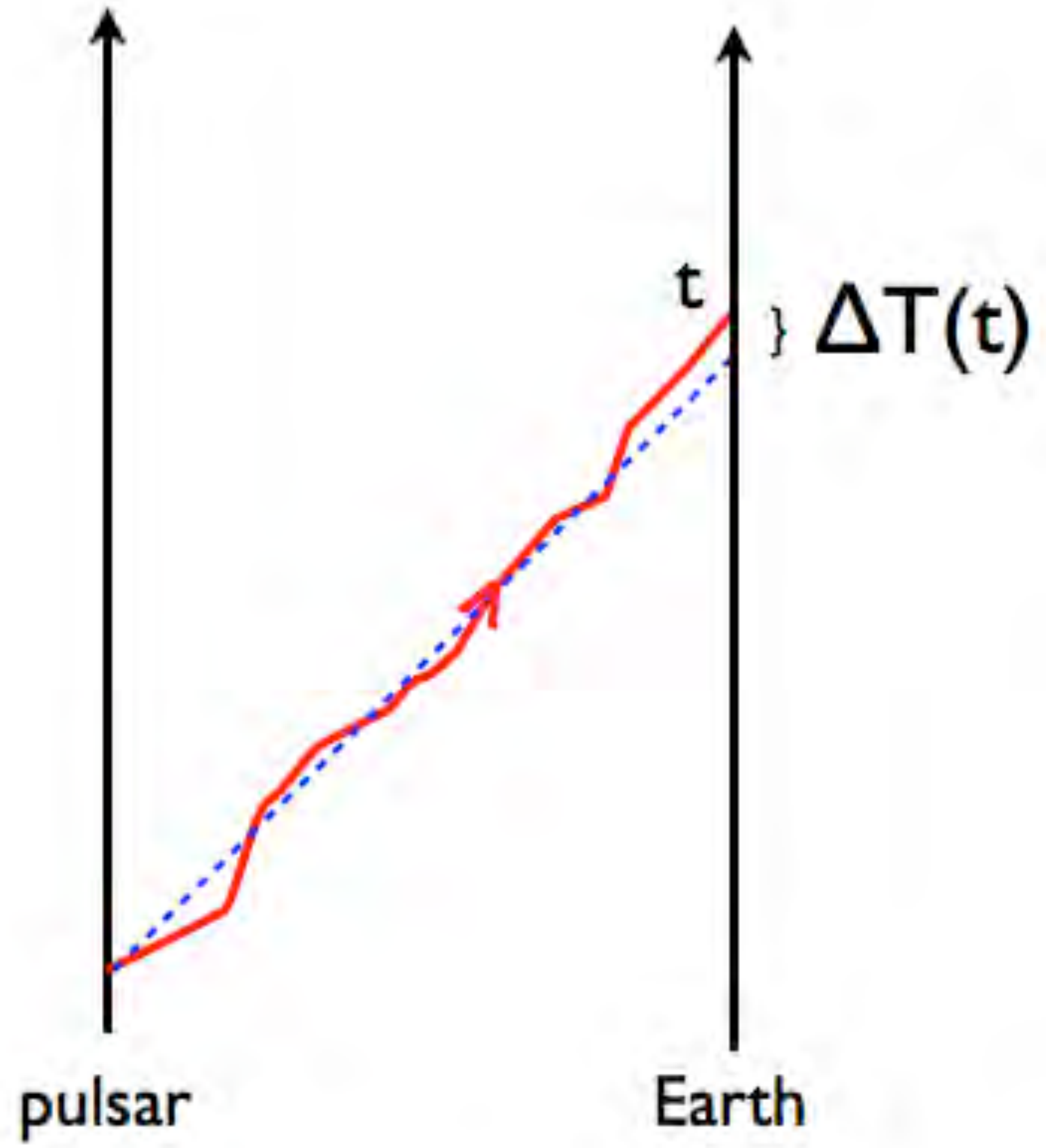




?

# Part 3. Decoding the Data

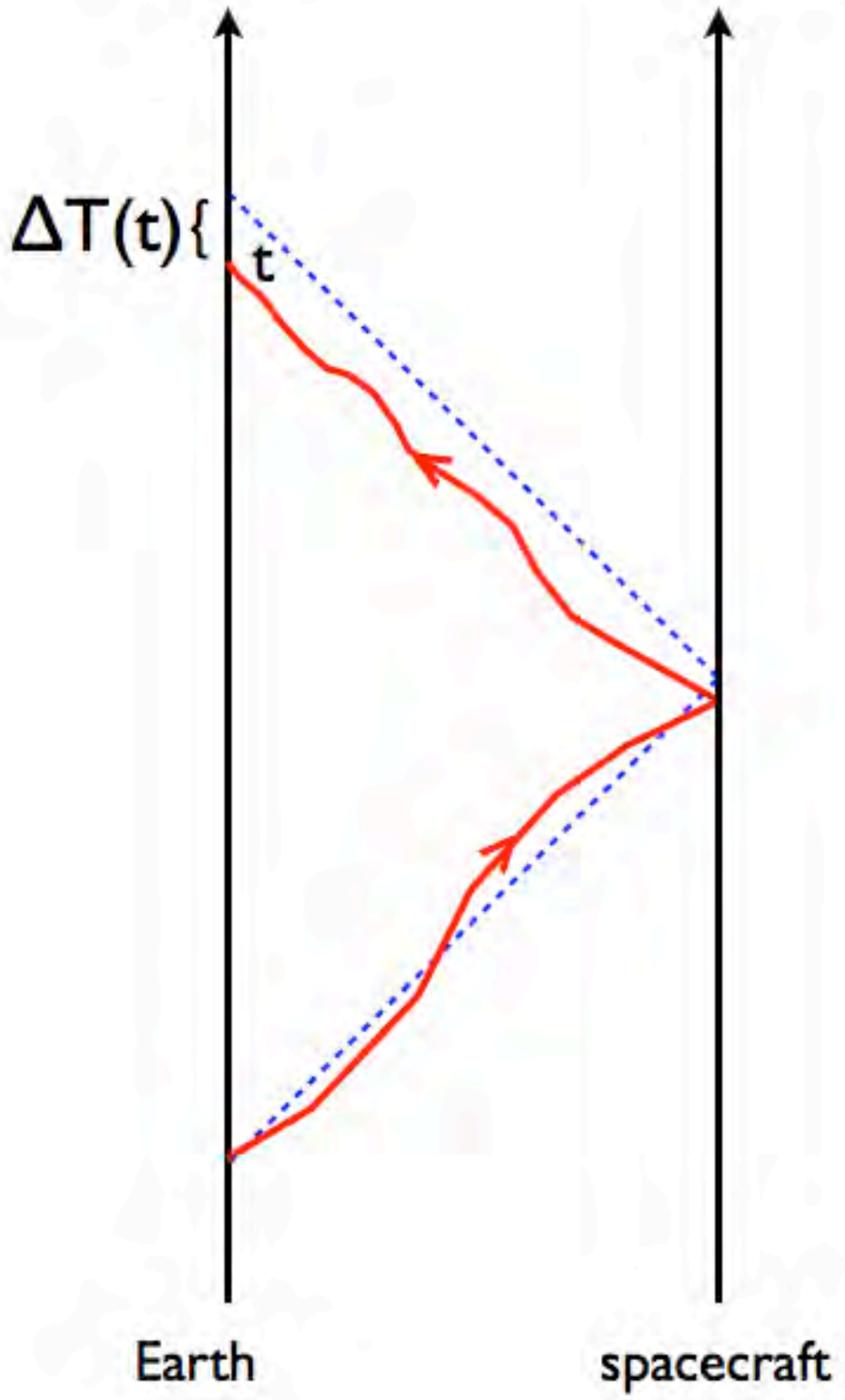
# Time of flight detectors

$$\Delta T(t)$$



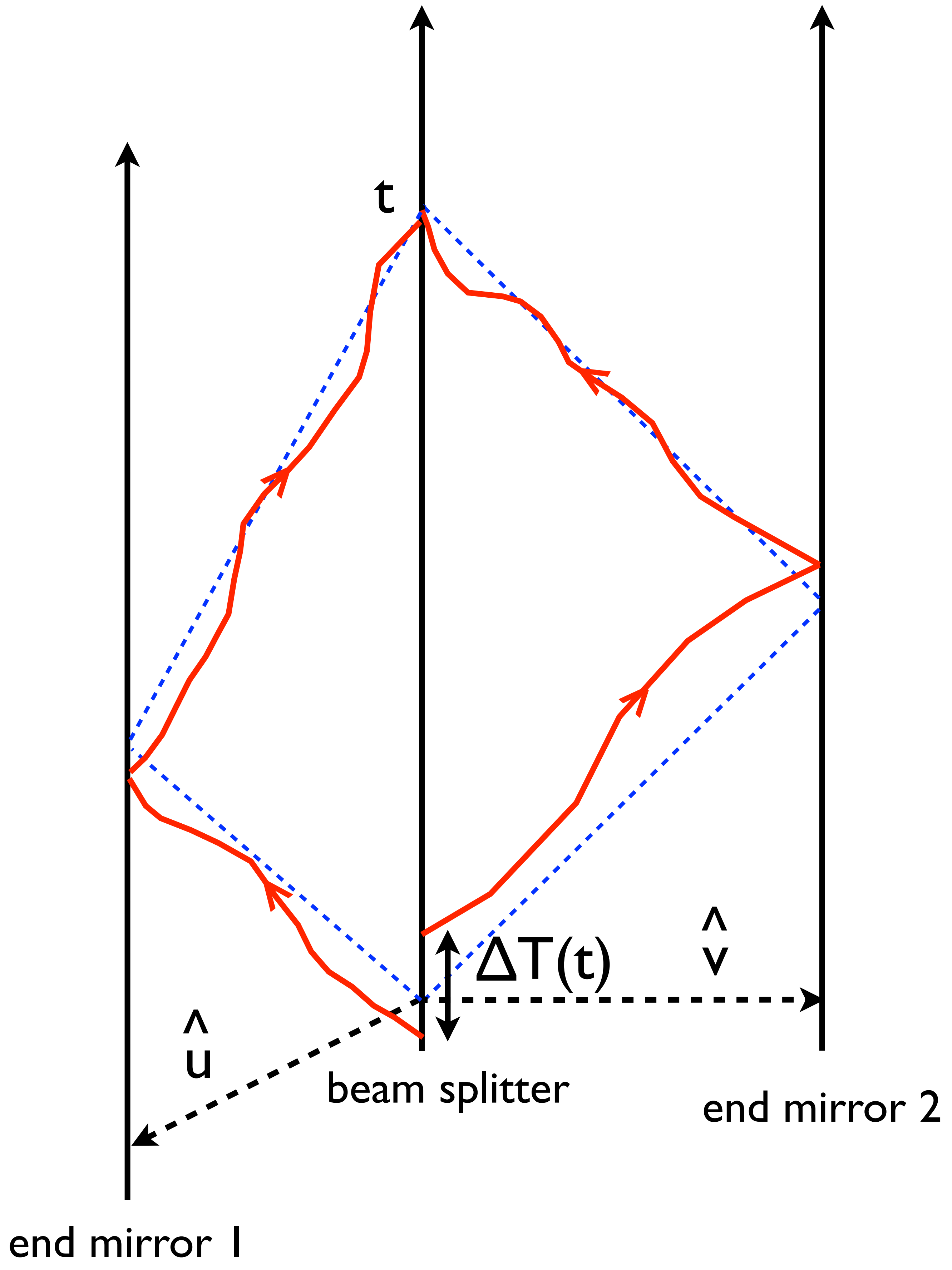

Pulsar Timing

$$\frac{\Delta\nu(t)}{\nu_0} = \frac{d\Delta T(t)}{dt}$$



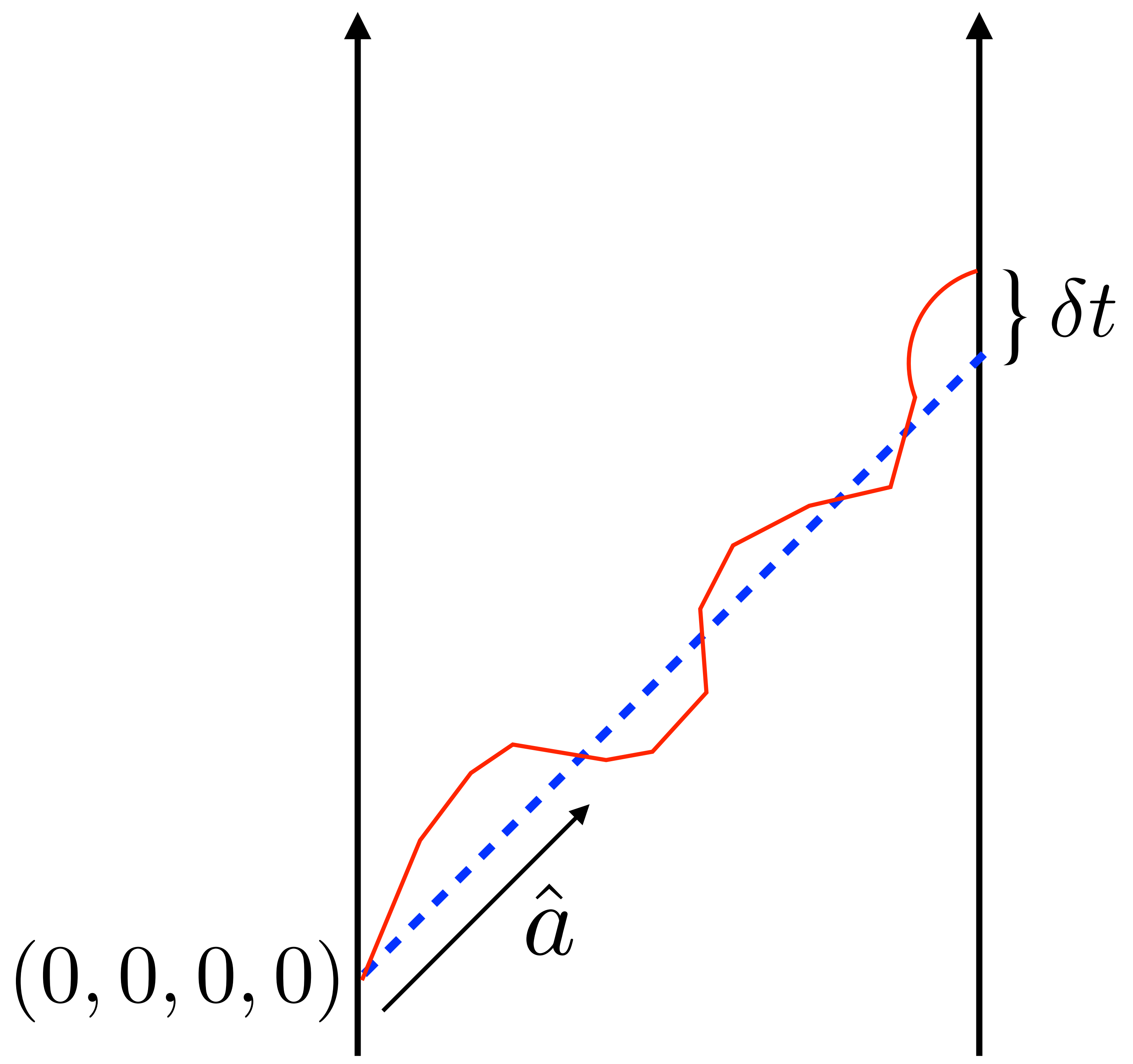

Spacecraft tracking

$$\Phi(t) = 2\pi\nu_0\Delta T(t)$$




Laser Interferometers

# Time of flight computed in TT gauge

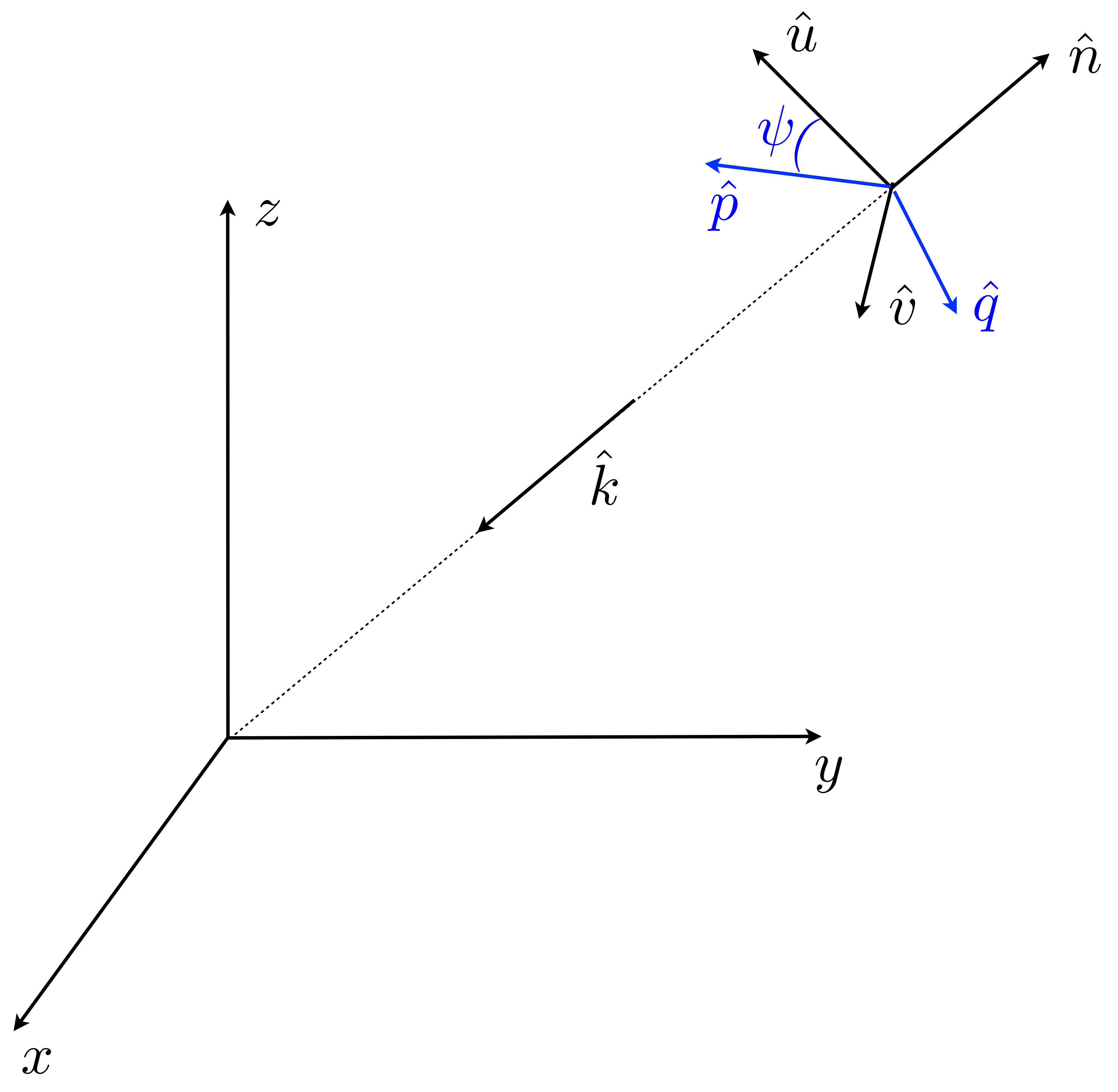



$$\Delta\tau_{12} = \frac{(\hat{a} \otimes \hat{a}) : \mathbf{H}[u_1, u_2]}{2(1 - \hat{k} \cdot \hat{a})} \qquad (u = k_\alpha x^\alpha)$$

Here $\hat{a}$ is a unit vector along the detector arm and $\hat{k}$ is the GW propagation direction

$$\mathbf{H}[u_1, u_2] = \int_{u_1}^{u_2} \mathbf{h}(u)\, du \qquad \mathbf{h} = h_+(u)\, \epsilon^+ + h_\times(u)\, \epsilon^\times$$

# General coordinate system




$$\hat{n} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$
$$\hat{u} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$
$$\hat{v} = \sin\phi\,\hat{x} - \cos\phi\,\hat{y}$$

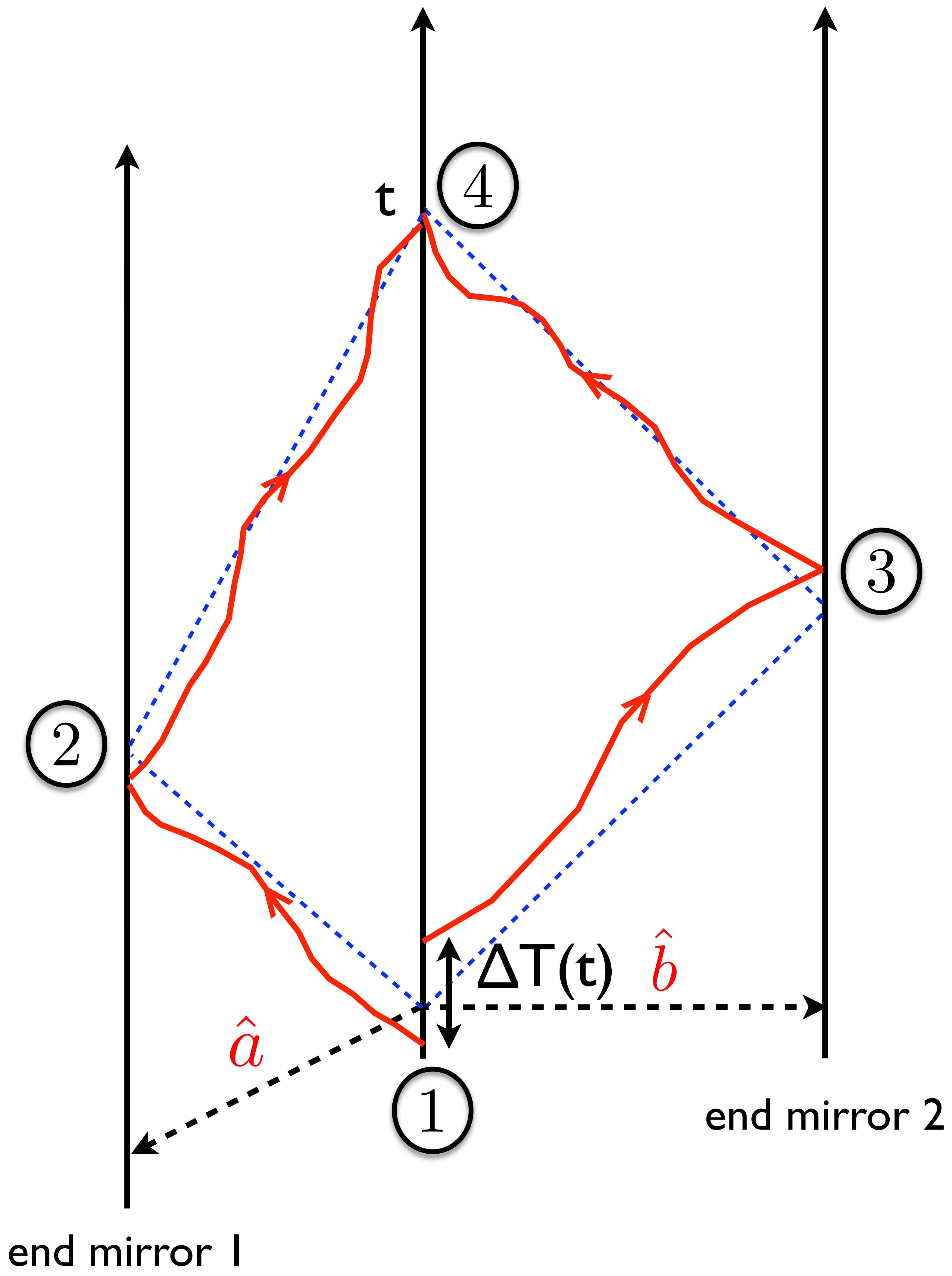
$$\mathbf{e}^{+} = \hat{u}\otimes\hat{u} - \hat{v}\otimes\hat{v}$$
$$\mathbf{e}^{\times} = \hat{u}\otimes\hat{v} + \hat{v}\otimes\hat{u}$$

$$\mathbf{h} = h_{+}\boldsymbol{\epsilon}^{+} + h_{\times}\boldsymbol{\epsilon}^{\times}$$

$$\begin{aligned}\epsilon^{+} &= \hat{p}\otimes\hat{p} - \hat{q}\otimes\hat{q} \\ &= \cos 2\psi\,\mathbf{e}^{+} - \sin 2\psi\,\mathbf{e}^{\times}\end{aligned}$$

$$\begin{aligned}\epsilon^{\times} &= \hat{p}\otimes\hat{q} + \hat{q}\otimes\hat{p} \\ &= \sin 2\psi\,\mathbf{e}^{+} + \cos 2\psi\,\mathbf{e}^{\times}\end{aligned}$$

# Example: Laser interferometer in the long wavelength limit




$$\Delta T(t) = \Delta\tau_{12} + \Delta\tau_{24} - \Delta\tau_{13} - \Delta\tau_{34}$$

$$h(t) \equiv \frac{\Delta T(t)}{2L} \approx \underbrace{\frac{1}{2}\left[\hat{a}\otimes\hat{a} - \hat{b}\otimes\hat{b}\right]}_{\text{Detector tensor}} : \mathbf{h}(t)$$

Detector tensor

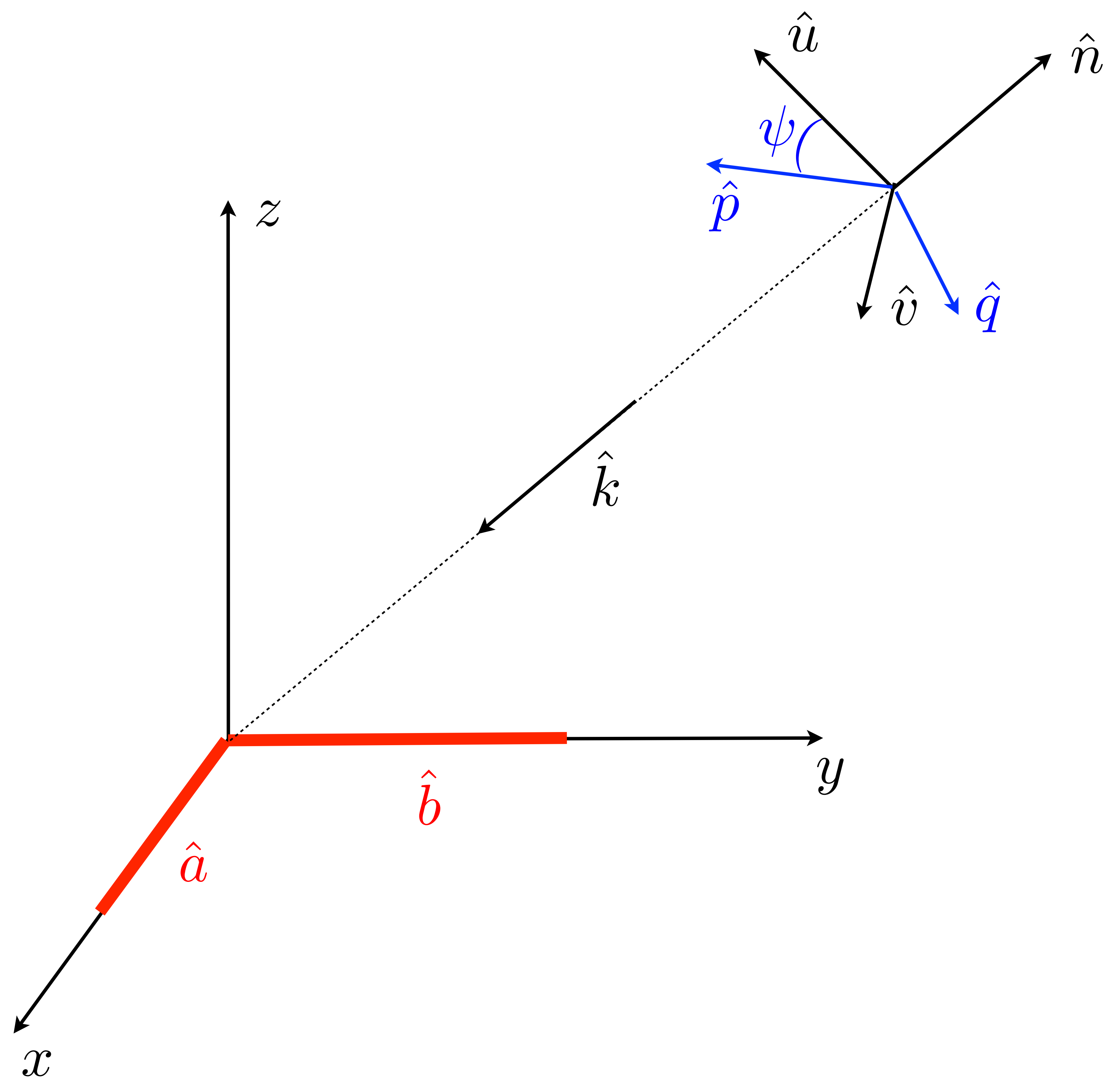
$$\mathbf{h}(t) = h_+(t)\boldsymbol{\epsilon}^+ + h_\times(t)\boldsymbol{\epsilon}^\times$$

Polarization tensors

# Antenna Pattern Functions

$$\hat{n} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$
$$\hat{u} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$
$$\hat{v} = \sin\phi\,\hat{x} - \cos\phi\,\hat{y}$$

$$\mathbf{e}^{+} = \hat{u}\otimes\hat{u} - \hat{v}\otimes\hat{v}$$
$$\mathbf{e}^{\times} = \hat{u}\otimes\hat{v} + \hat{v}\otimes\hat{u}$$




$$(\hat{a}\otimes\hat{a}) : \mathbf{e}^{+} = \cos^2\theta\cos^2\phi - \sin^2\phi$$

$$(\hat{a}\otimes\hat{a}) : \mathbf{e}^{\times} = \cos\theta\sin 2\phi$$

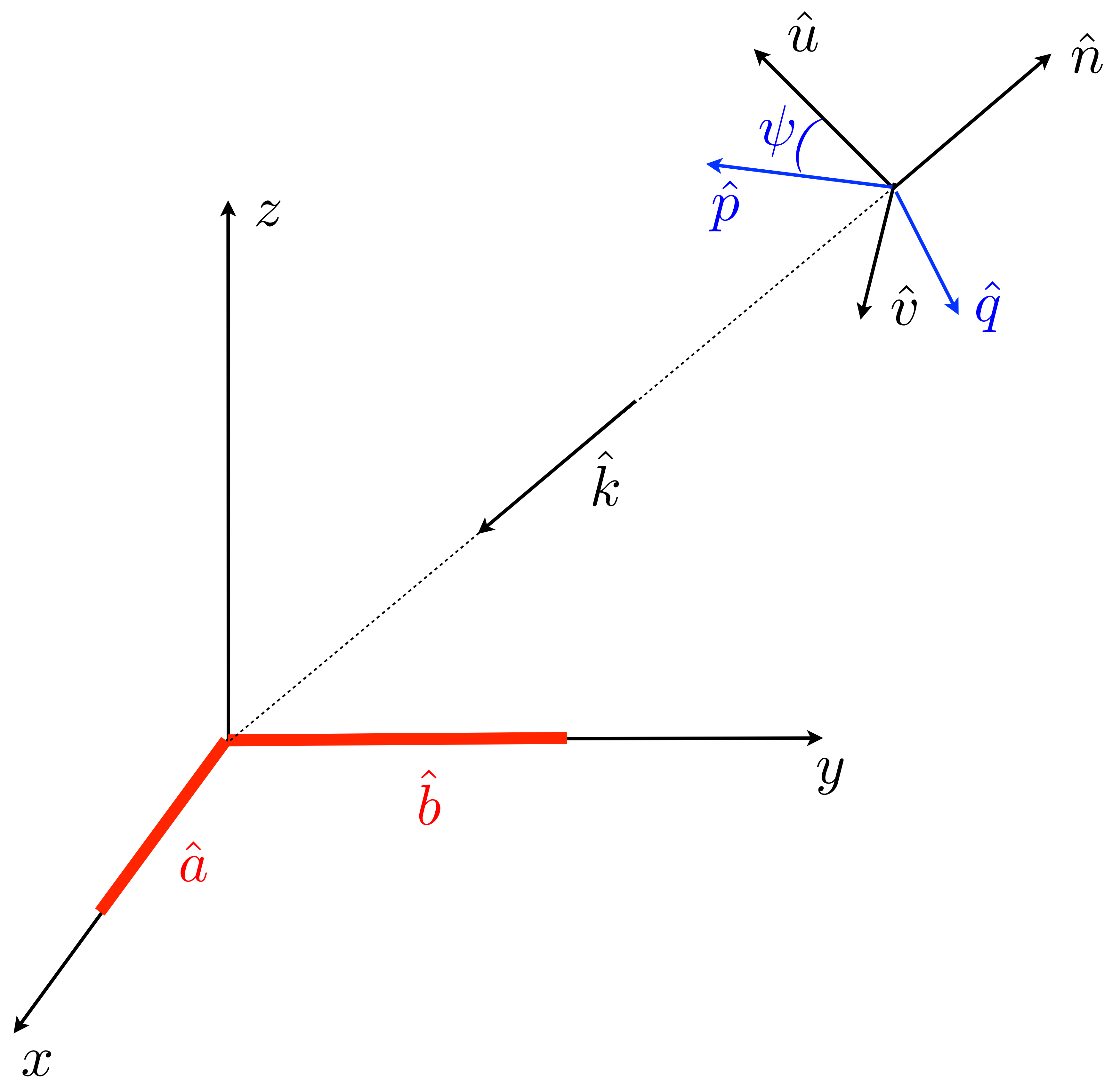
$$(\hat{b}\otimes\hat{b}) : \mathbf{e}^{+} = \cos^2\theta\sin^2\phi - \cos^2\phi$$

$$(\hat{b}\otimes\hat{b}) : \mathbf{e}^{\times} = -\cos\theta\sin 2\phi$$

# Antenna Pattern Functions

$$\hat{n} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$
$$\hat{u} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$
$$\hat{v} = \sin\phi\,\hat{x} - \cos\phi\,\hat{y}$$

$$\mathbf{e}^{+} = \hat{u}\otimes\hat{u} - \hat{v}\otimes\hat{v}$$
$$\mathbf{e}^{\times} = \hat{u}\otimes\hat{v} + \hat{v}\otimes\hat{u}$$



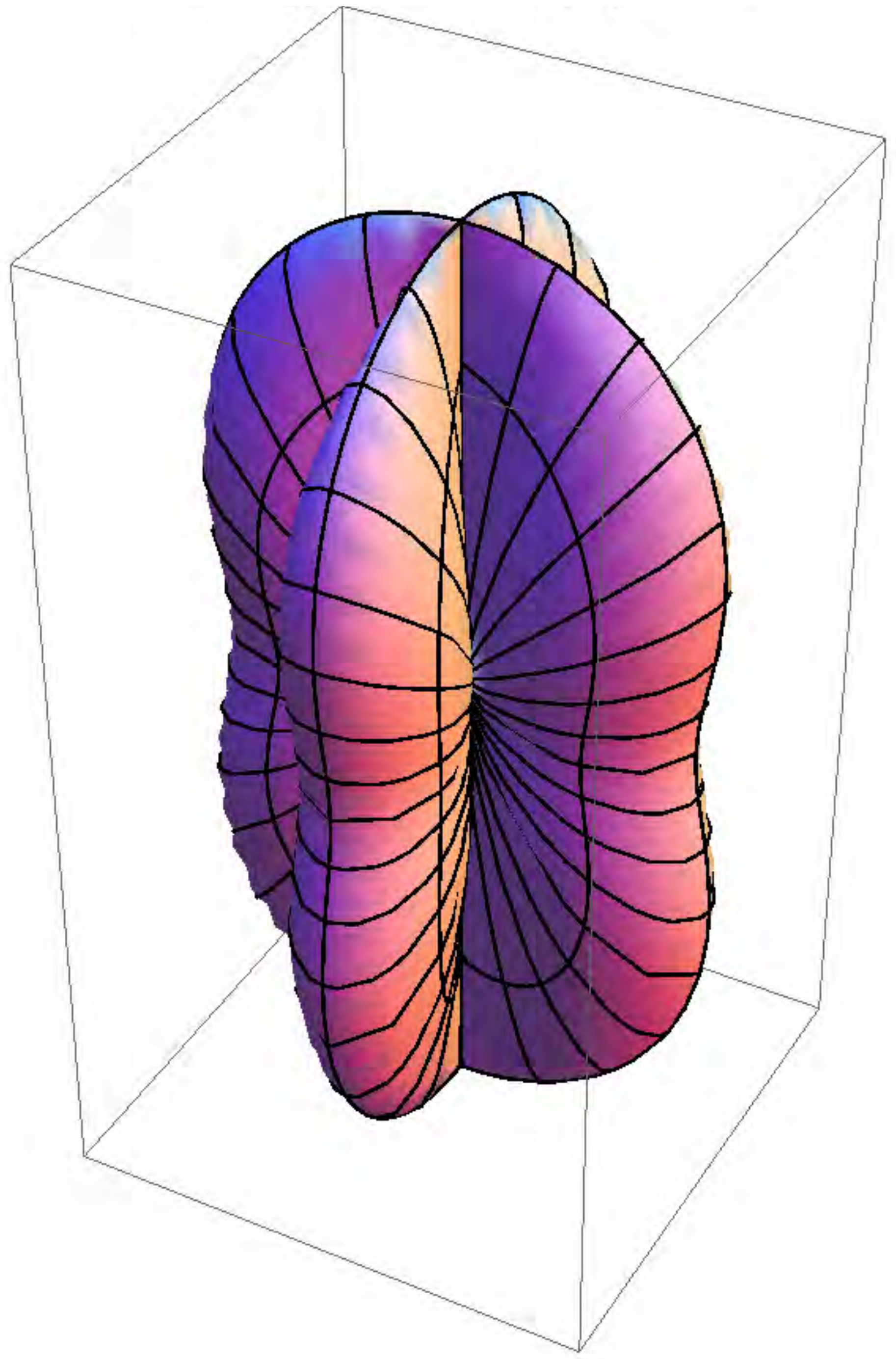

$$h = F^{+}h_{+} + F^{\times}h_{\times}$$

$$F^{+} = \frac{1}{2}(\hat{a}\otimes\hat{a} - \hat{b}\otimes\hat{b}) : \epsilon^{+}$$
$$= \frac{1}{2}(1+\cos^2\theta)\cos(2\phi)\cos 2\psi - \cos\theta\sin 2\phi\sin 2\psi$$

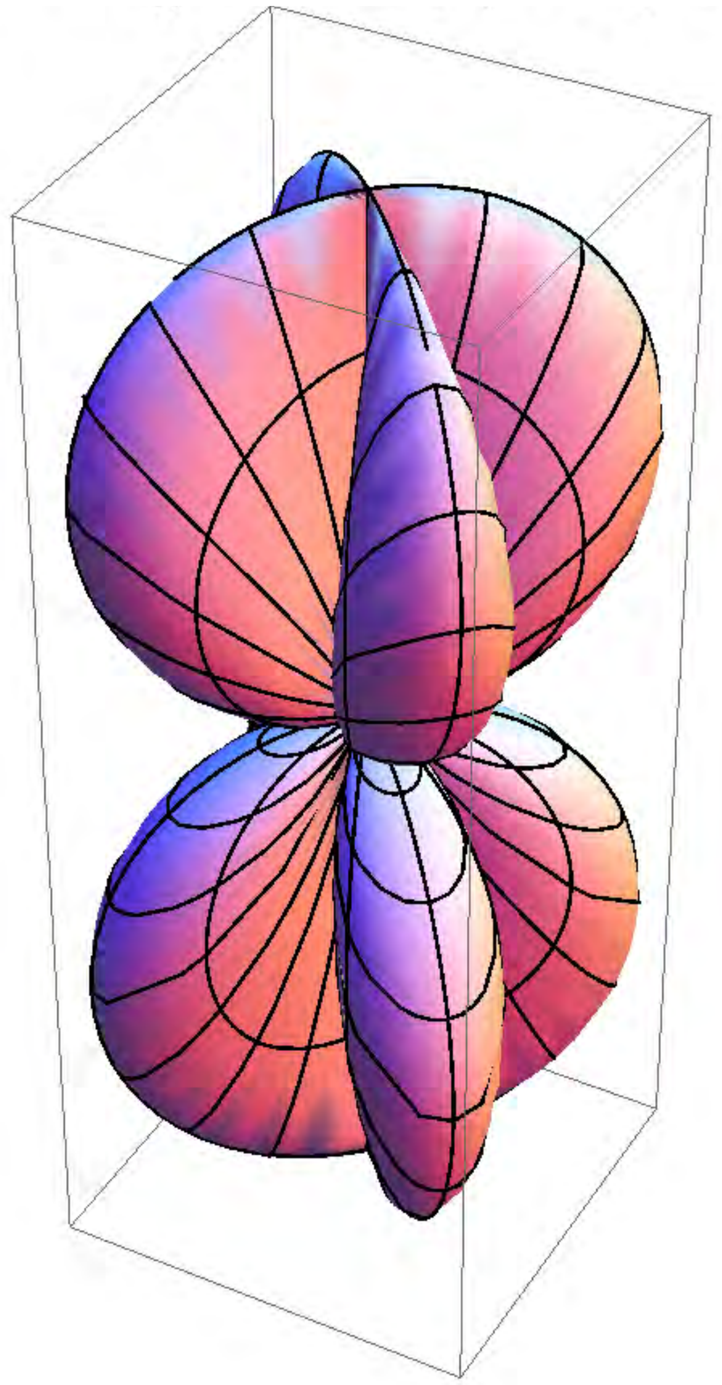
$$F^{\times} = \frac{1}{2}(\hat{a}\otimes\hat{a} - \hat{b}\otimes\hat{b}) : \epsilon^{\times}$$
$$= \frac{1}{2}(1+\cos^2\theta)\cos(2\phi)\sin 2\psi + \cos\theta\sin 2\phi\cos 2\psi$$

# Antenna Pattern Functions

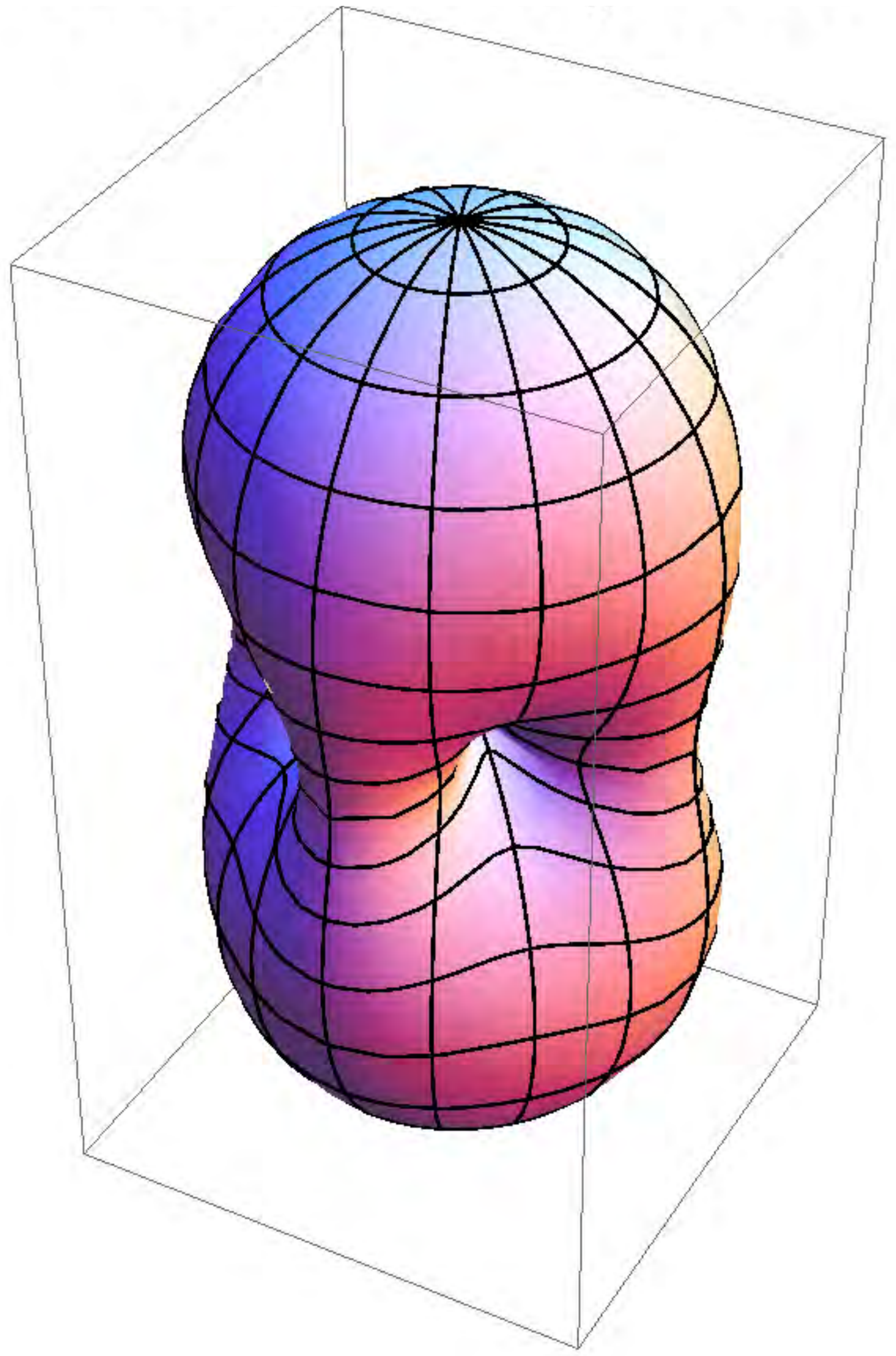
$F_+$



$F_\times$

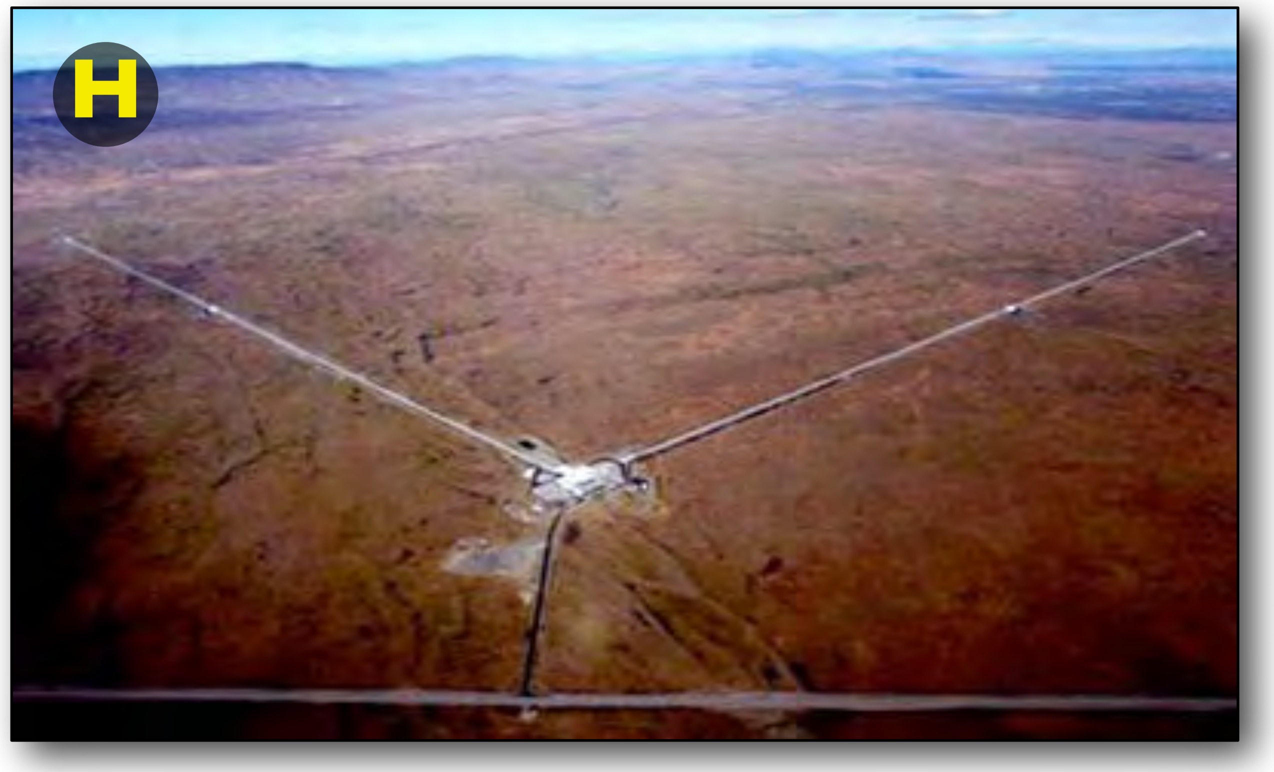


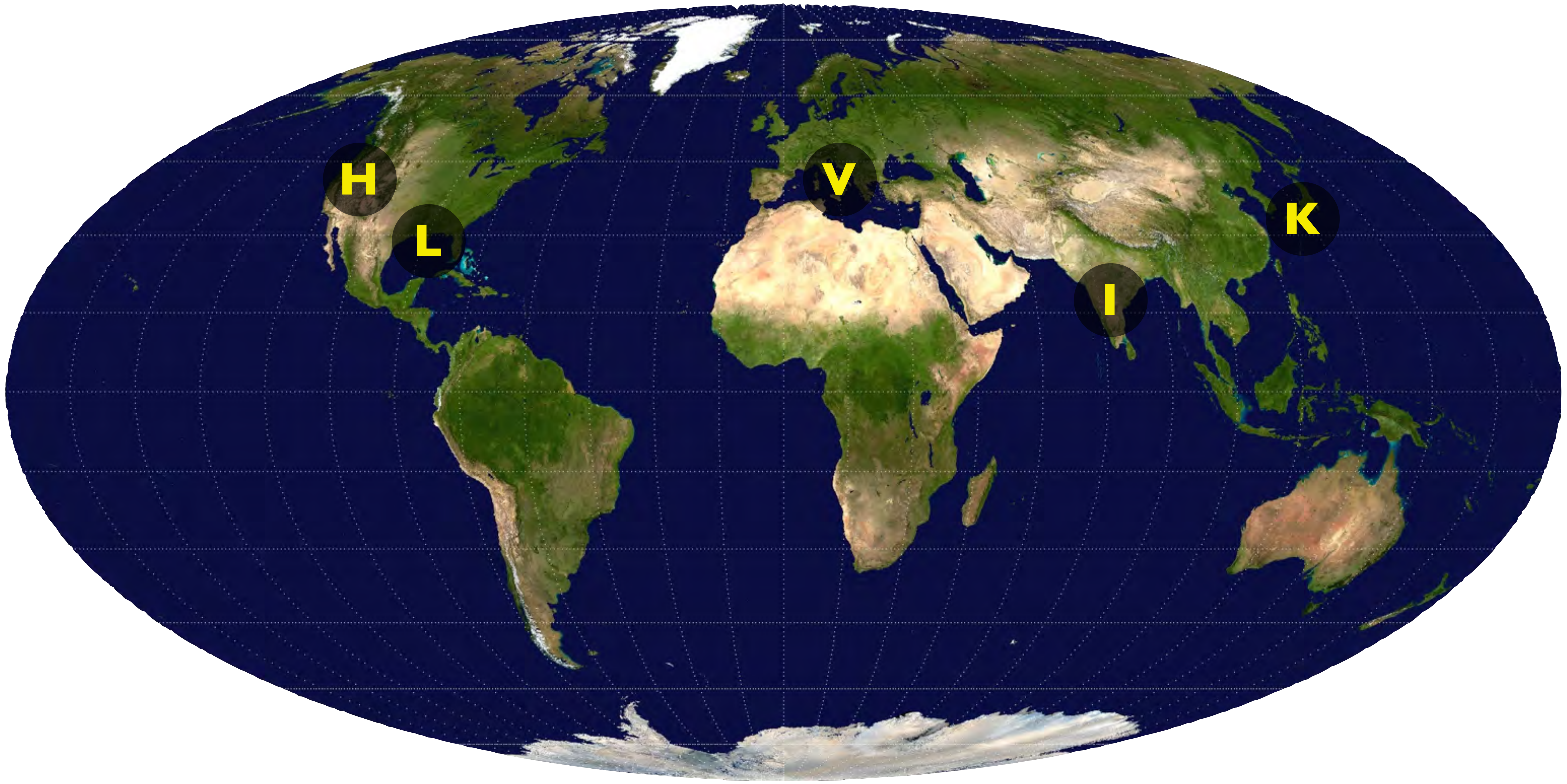
$F = \sqrt{F_+^2 + F_\times^2}$



Polarization averaged

# Terrestrial Network

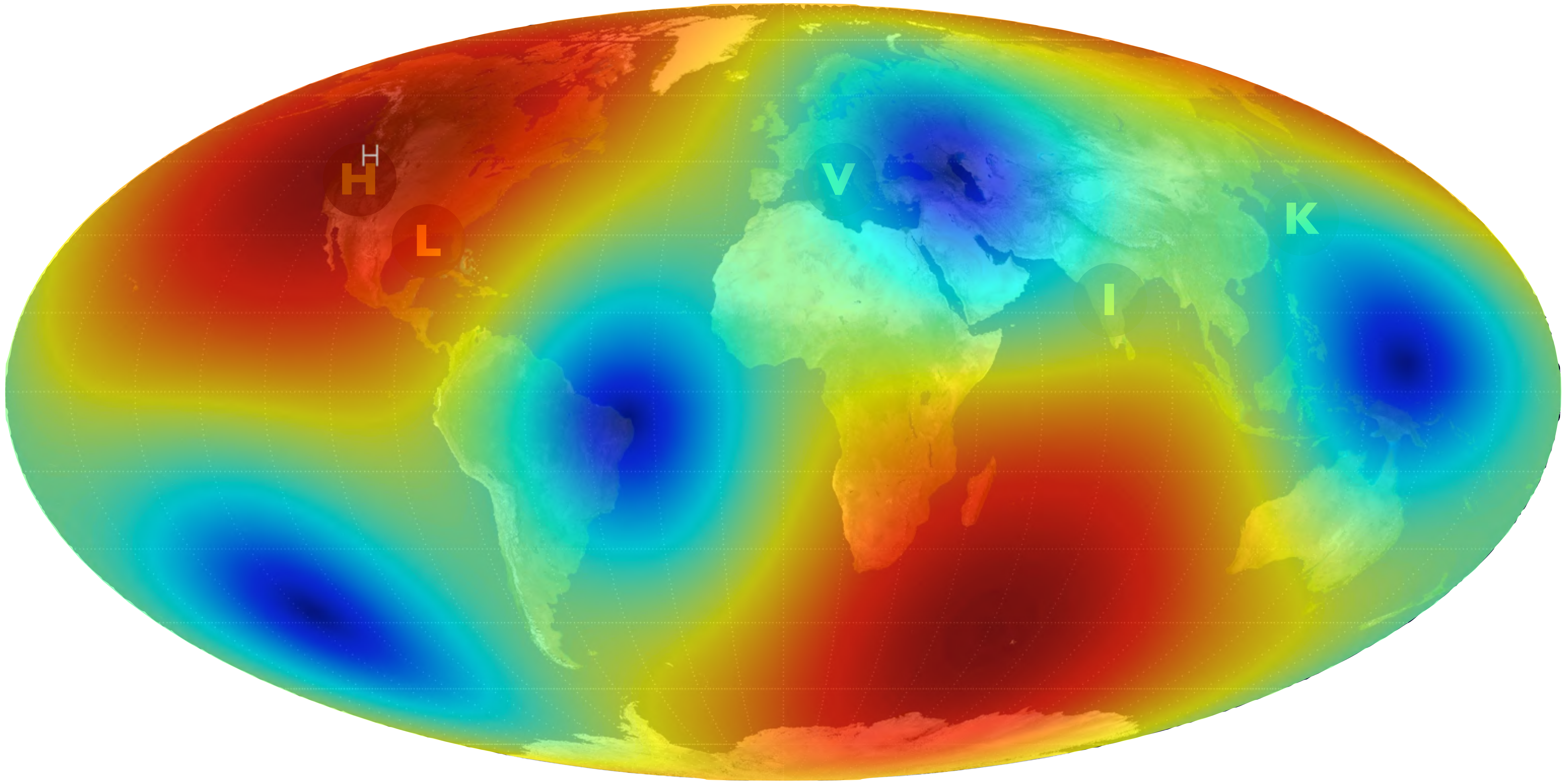






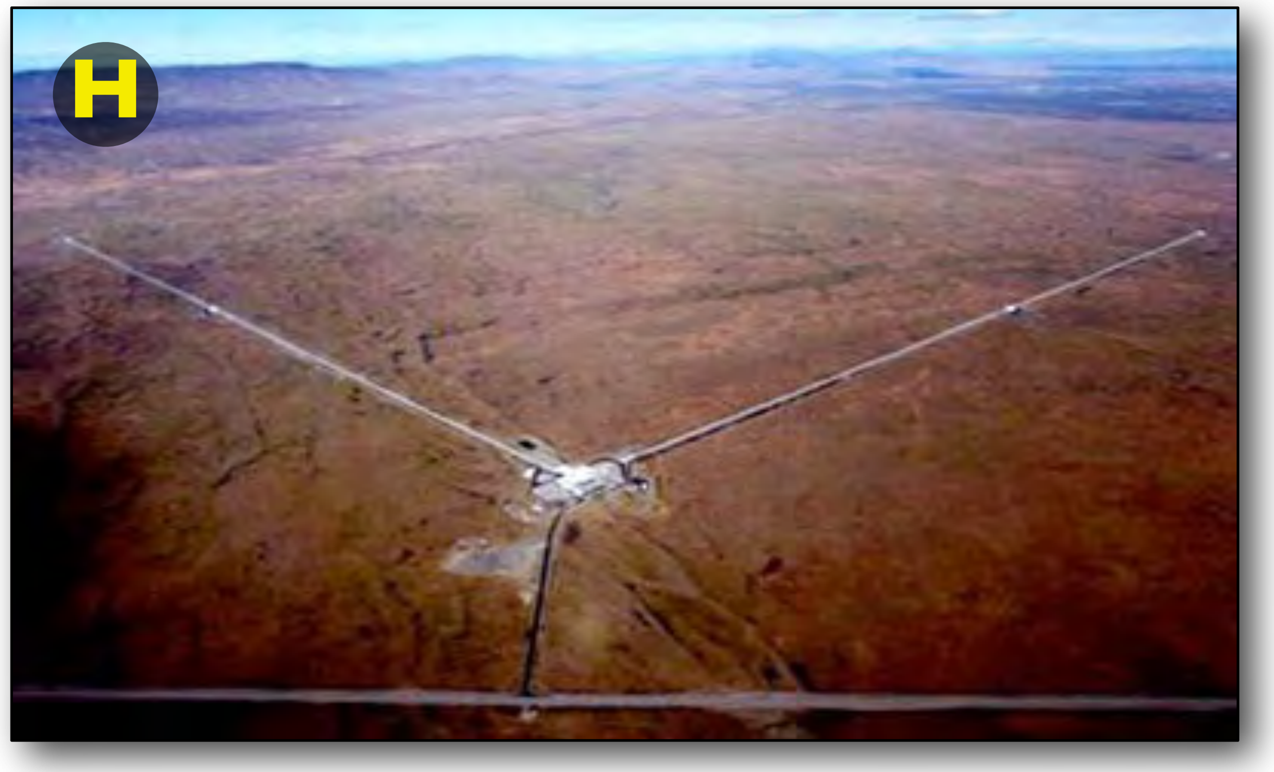
# Terrestrial Network

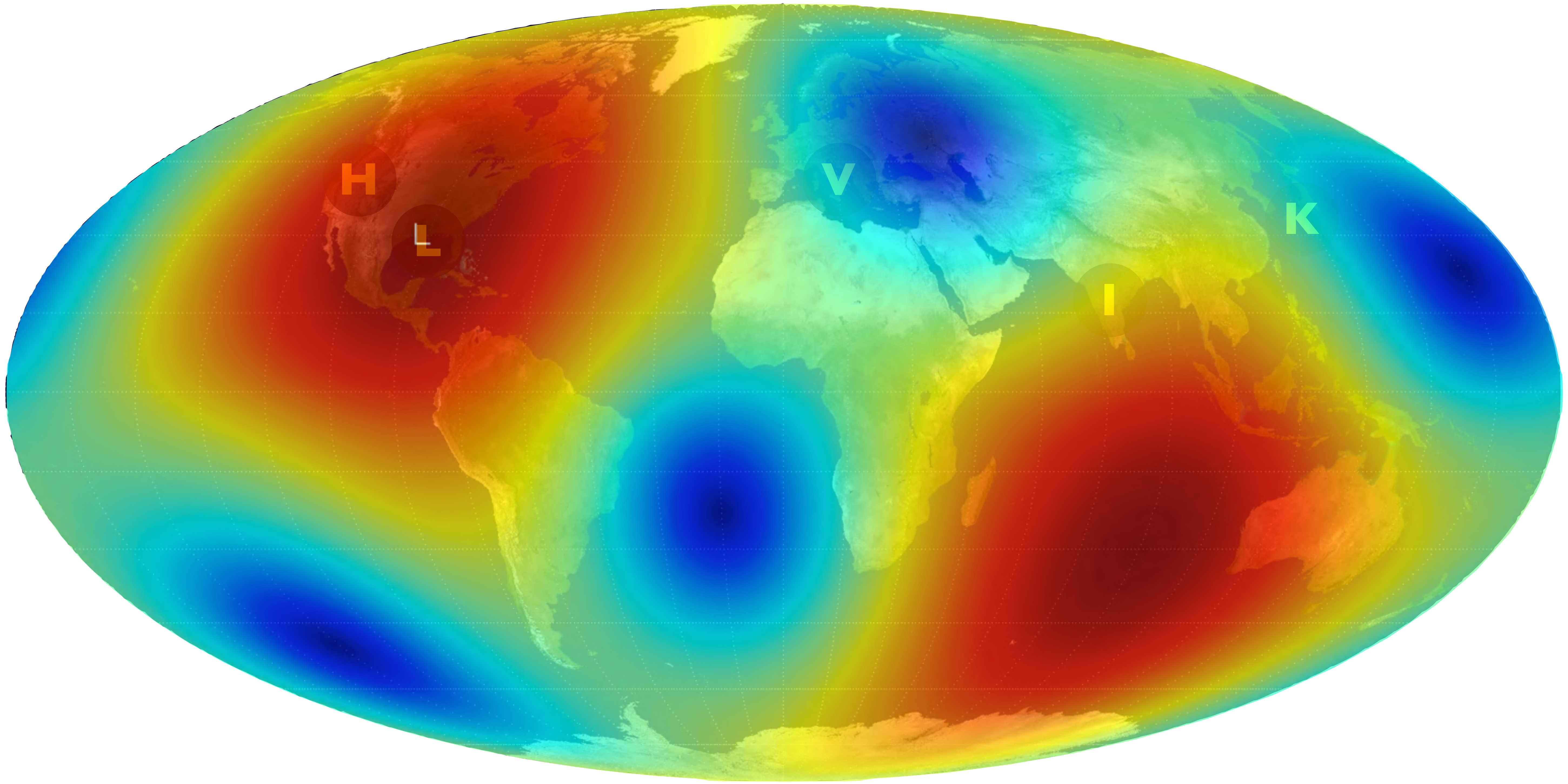







# Terrestrial Network

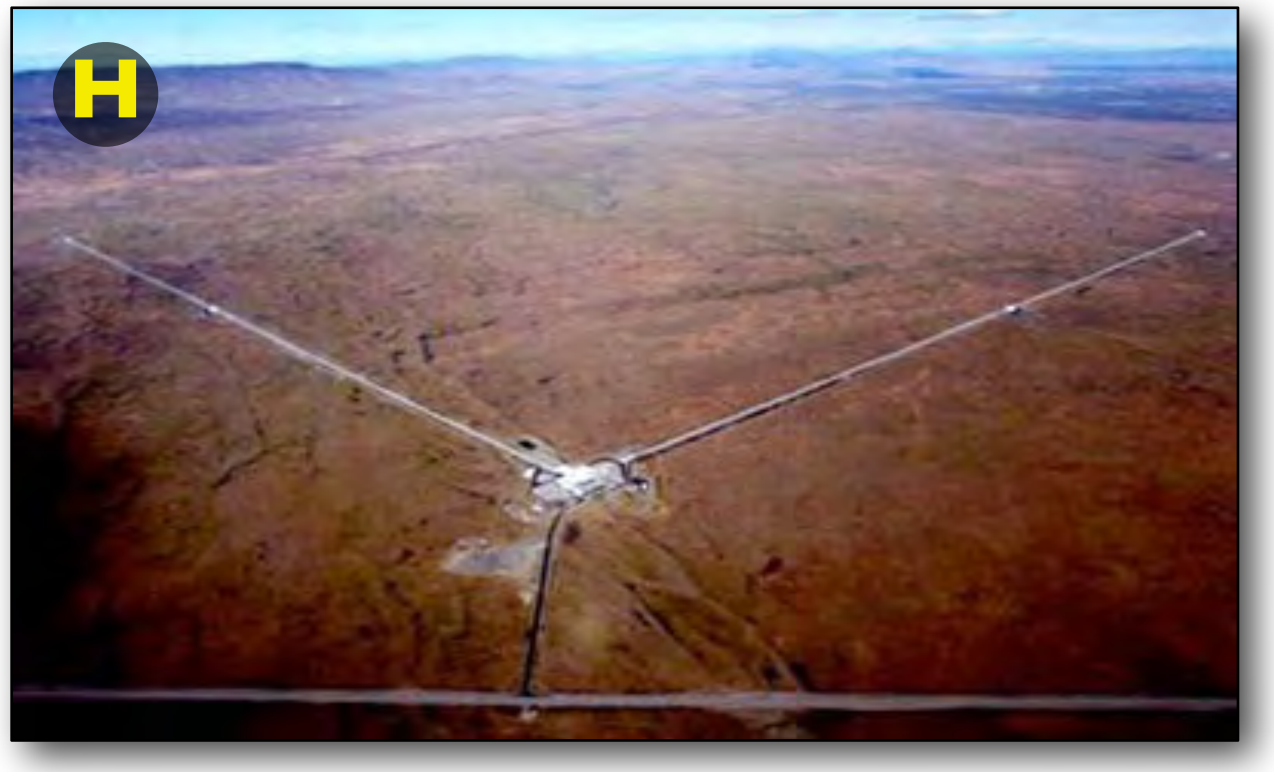


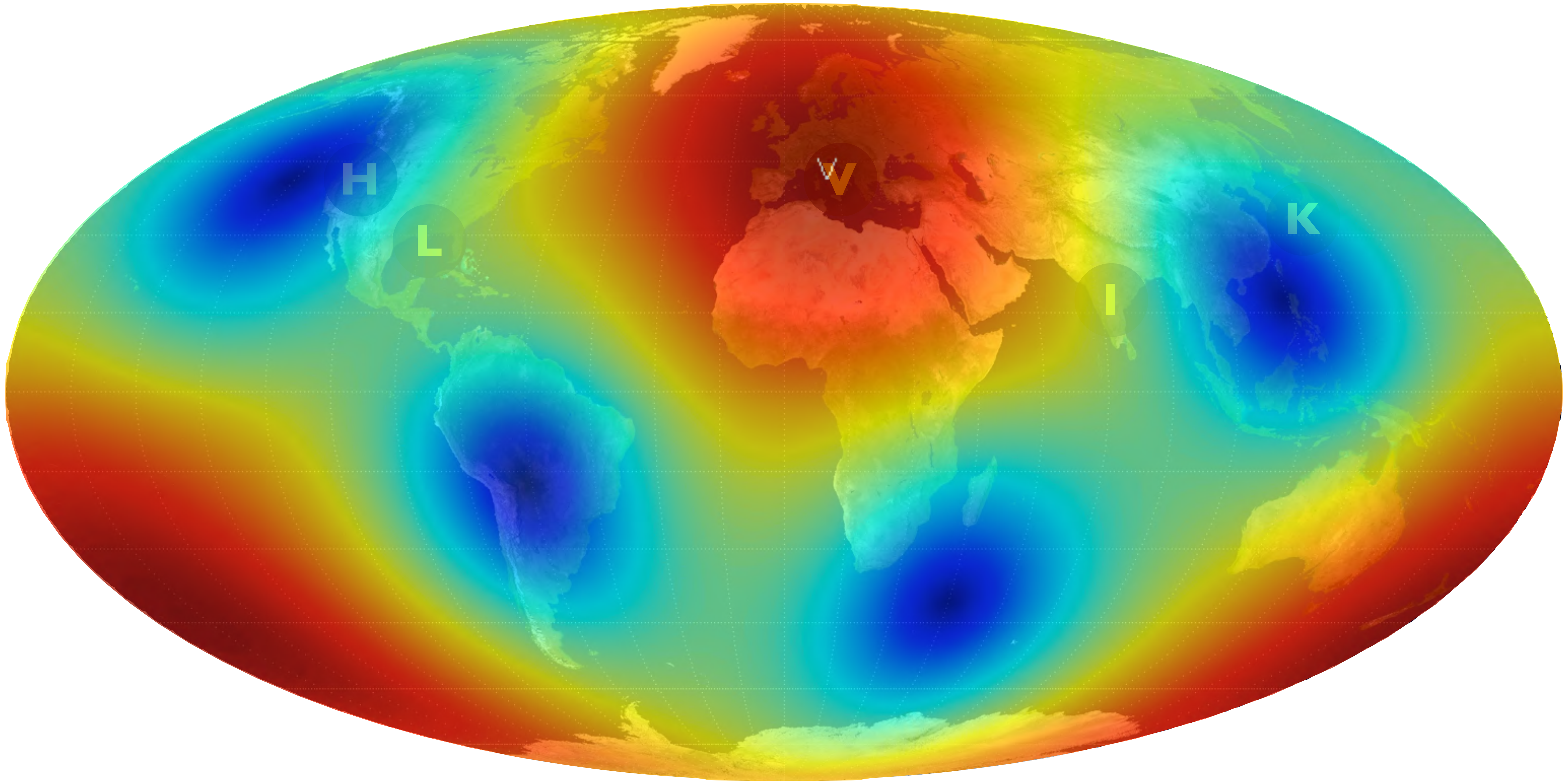




# Terrestrial Network

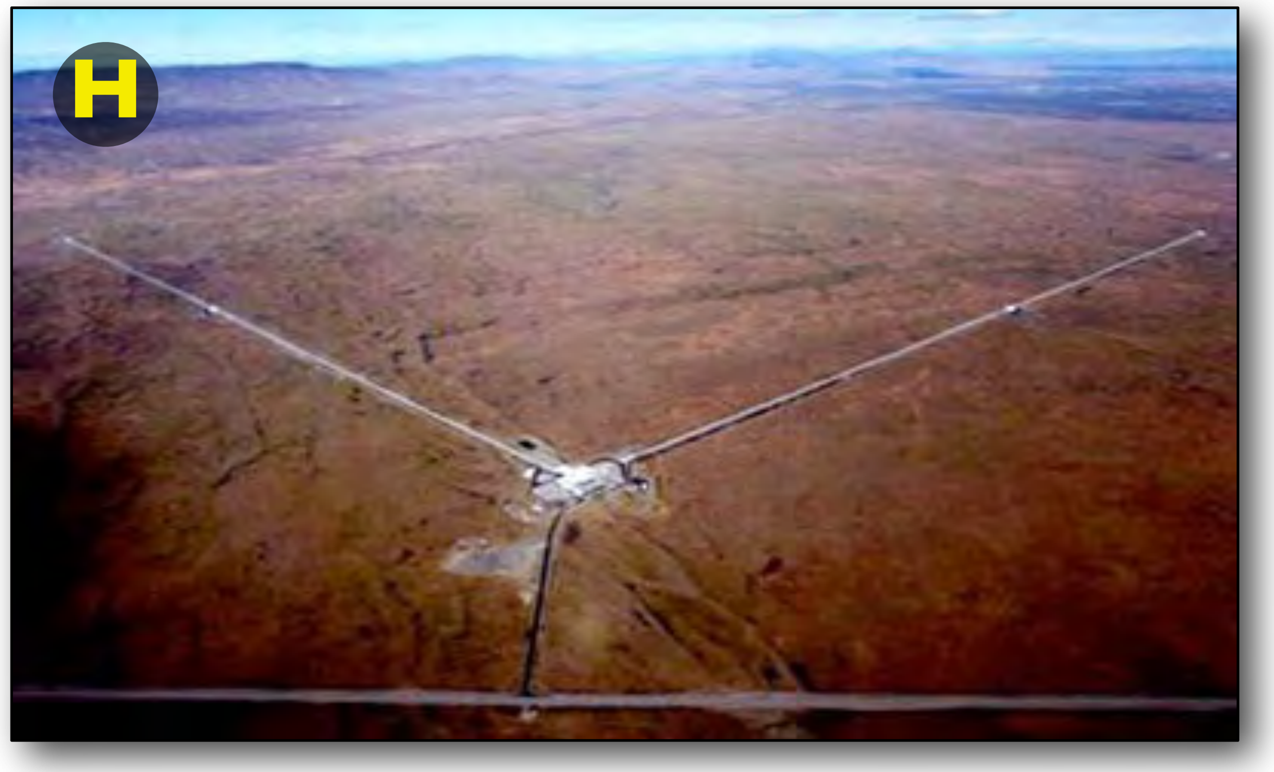


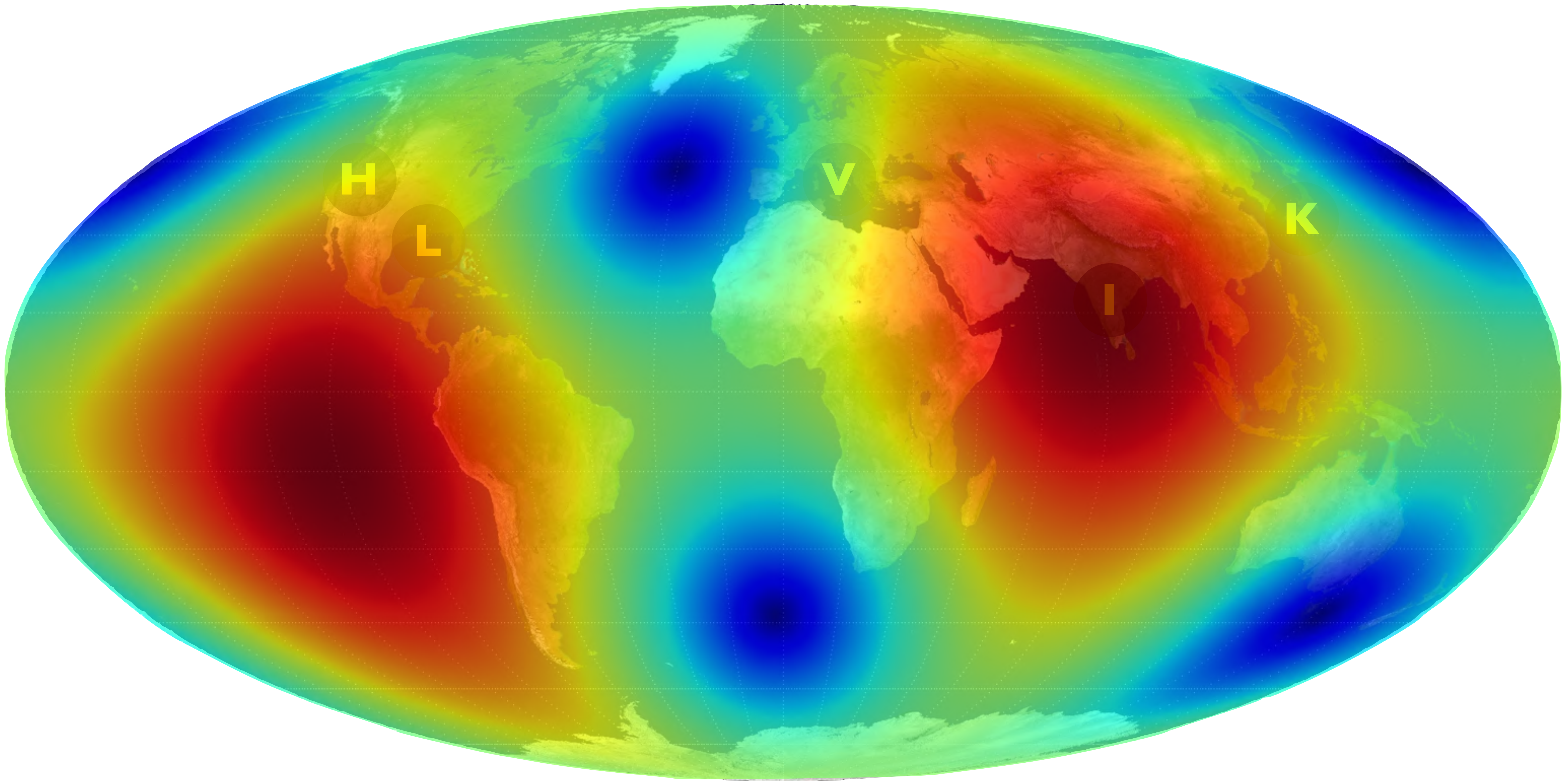




# Terrestrial Network

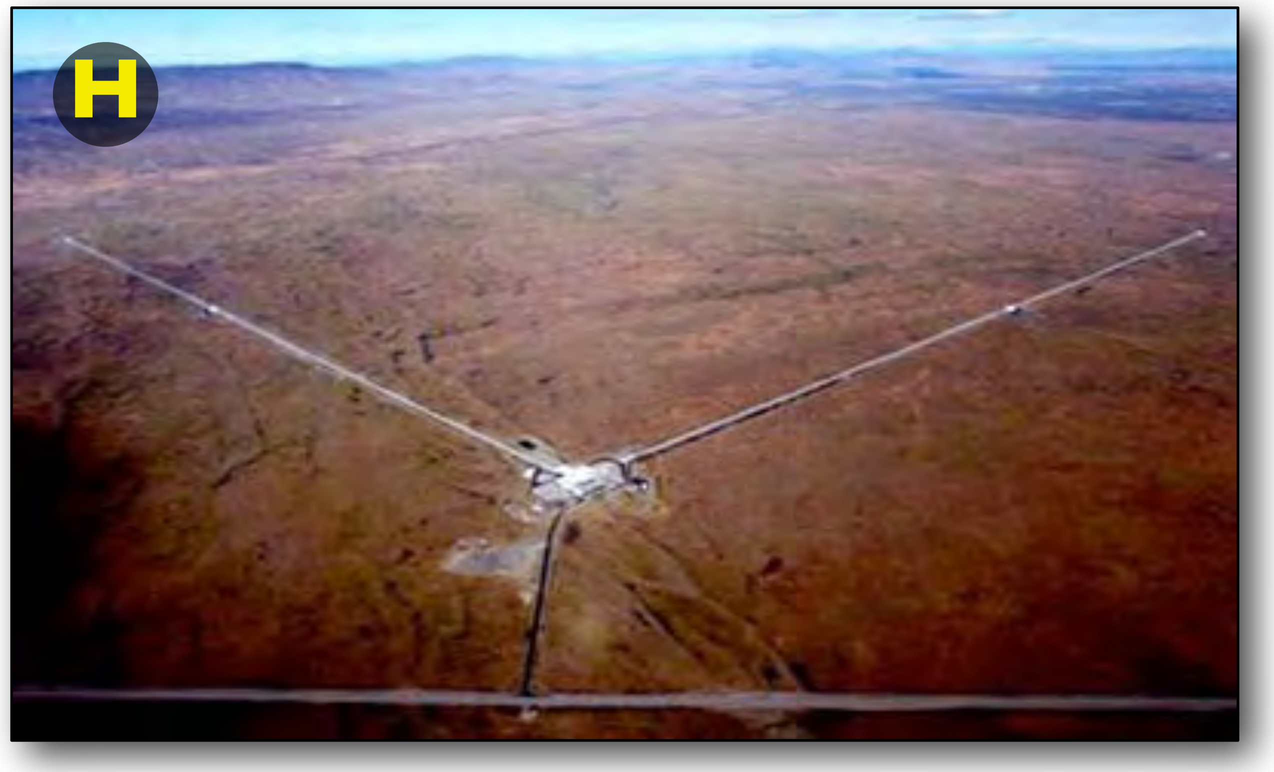


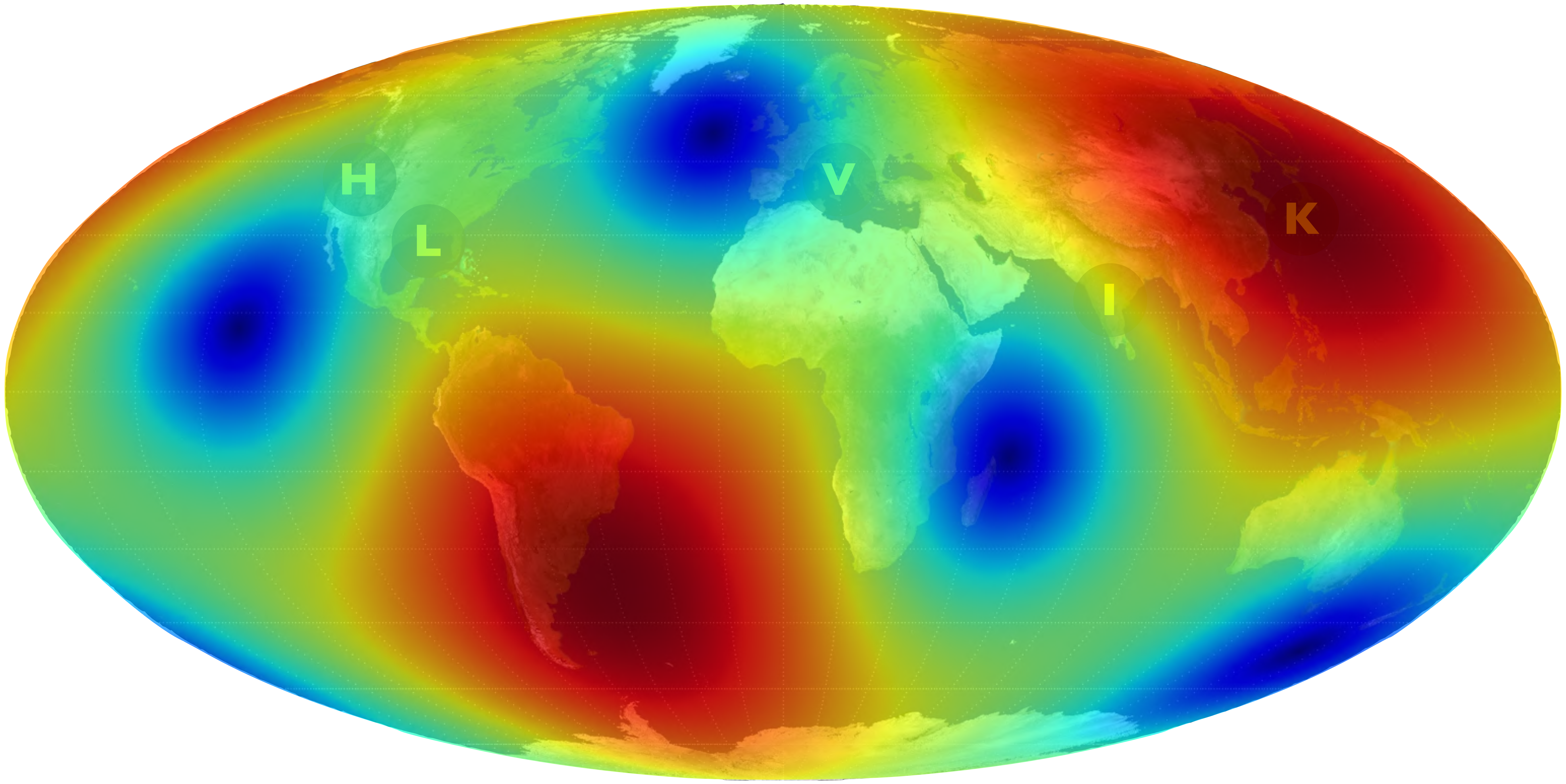







# Terrestrial Network

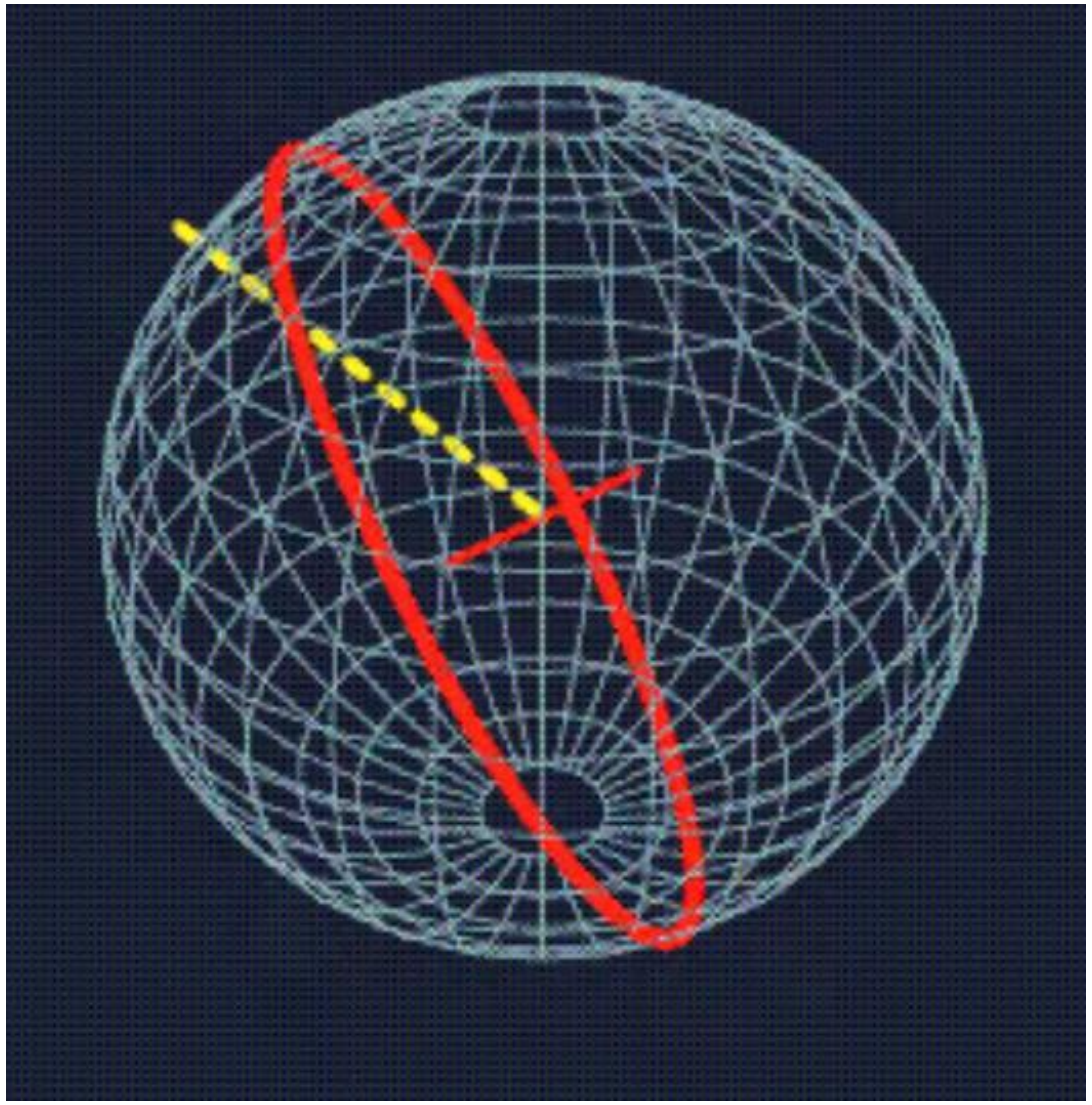


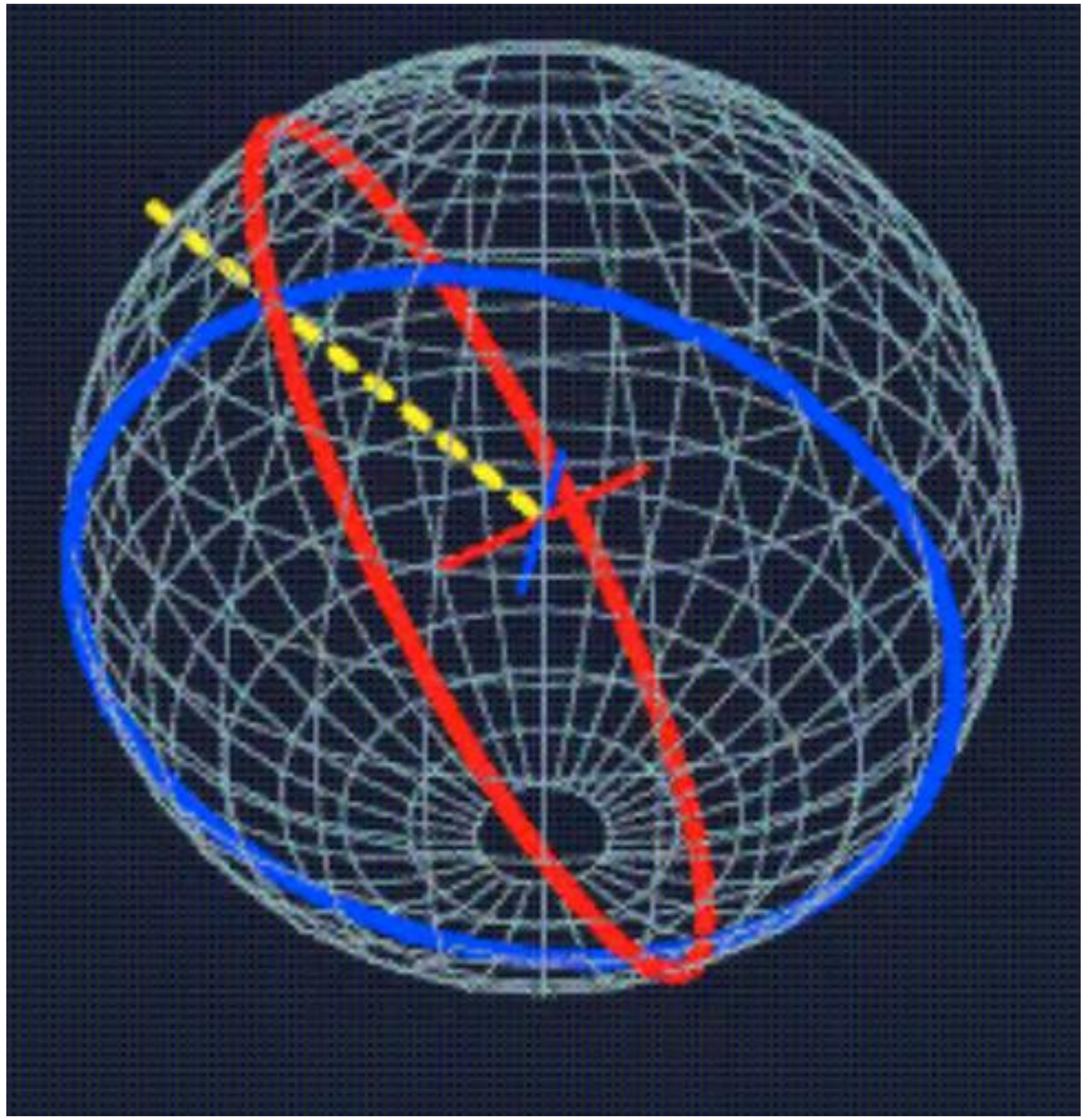




# Time of Arrival Triangulation

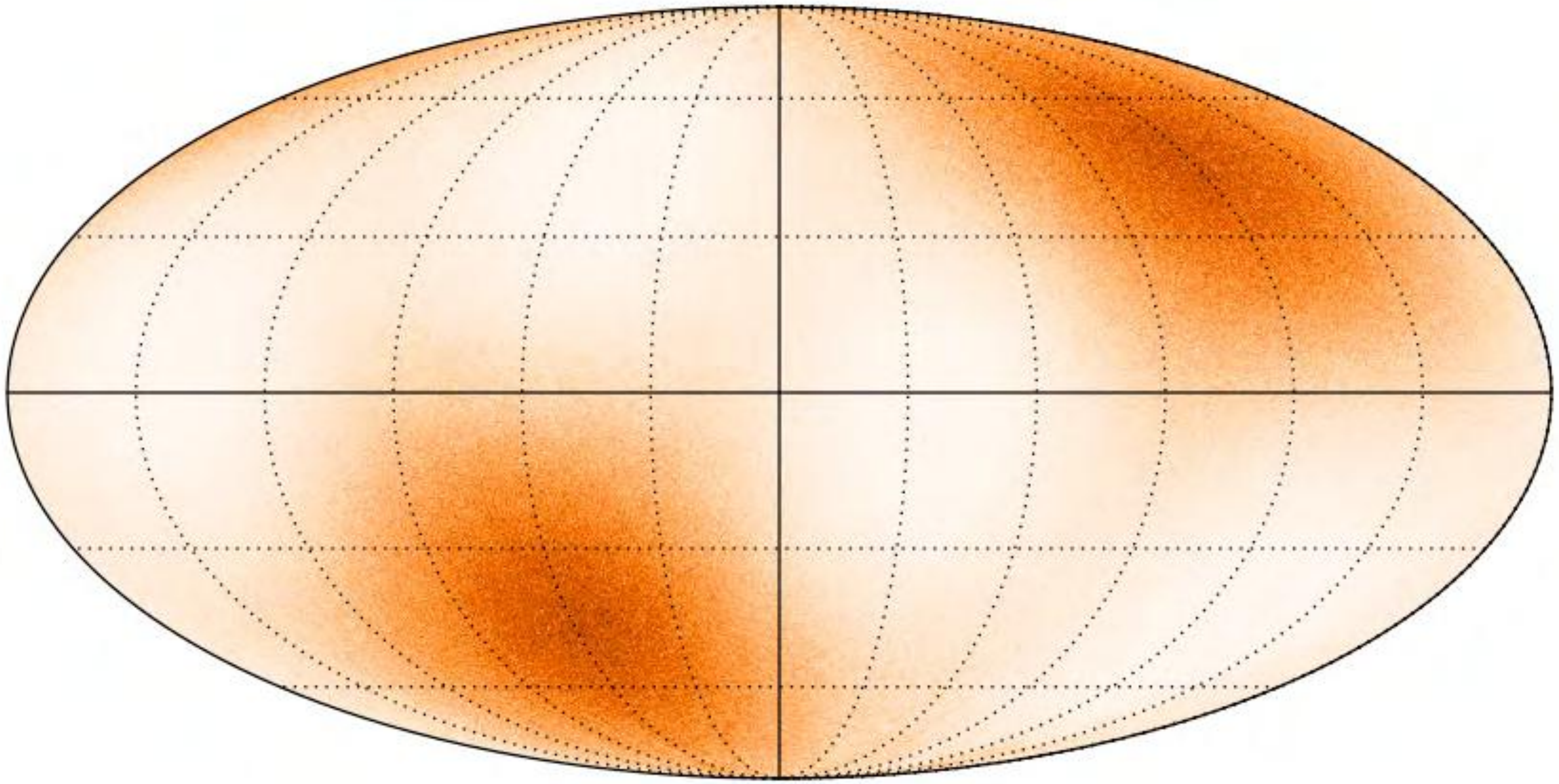


LIGO Hanford + LIGO Livingston



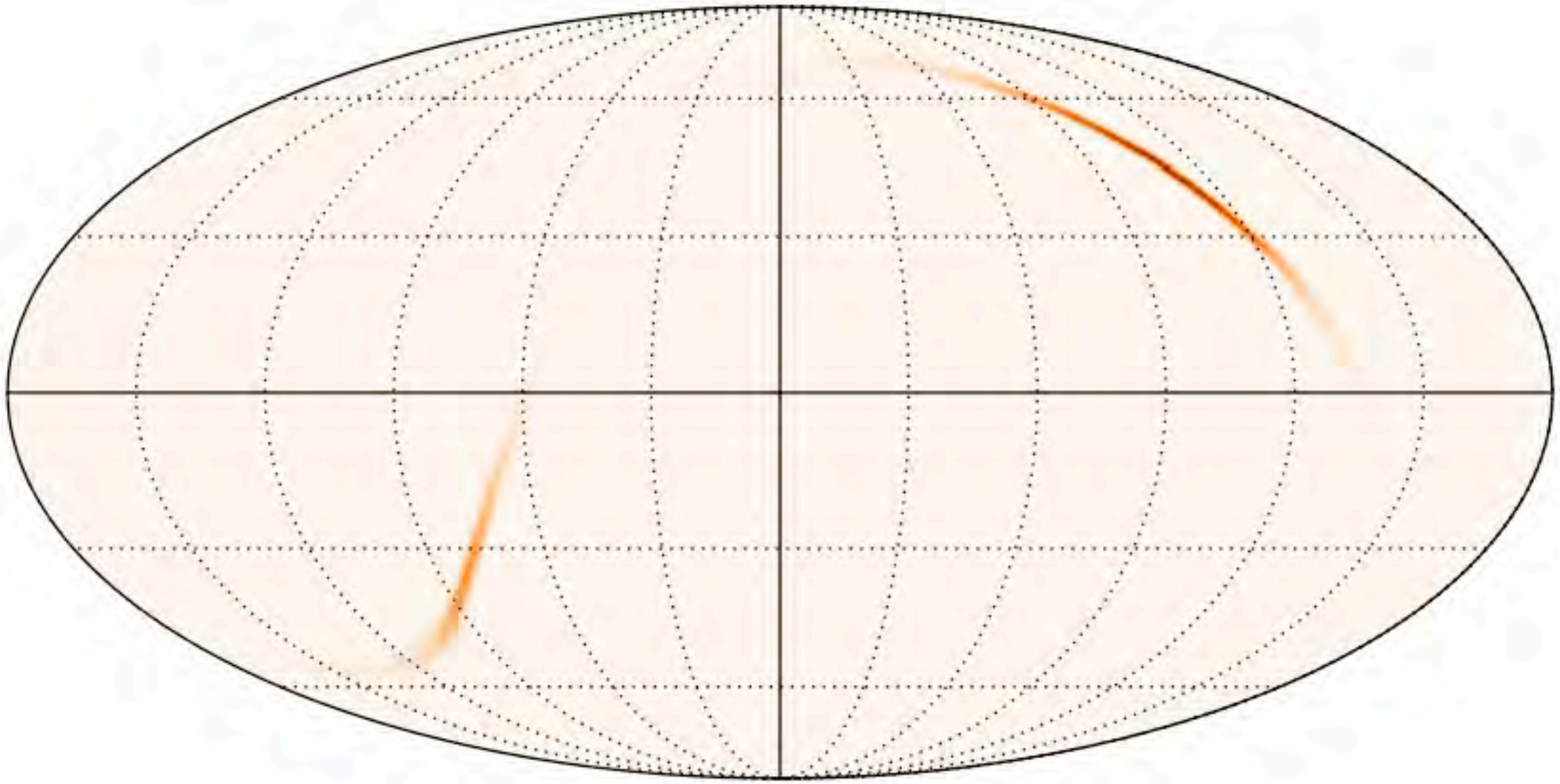
LIGO Hanford + LIGO Livingston + Virgo

# Triangulating the Source



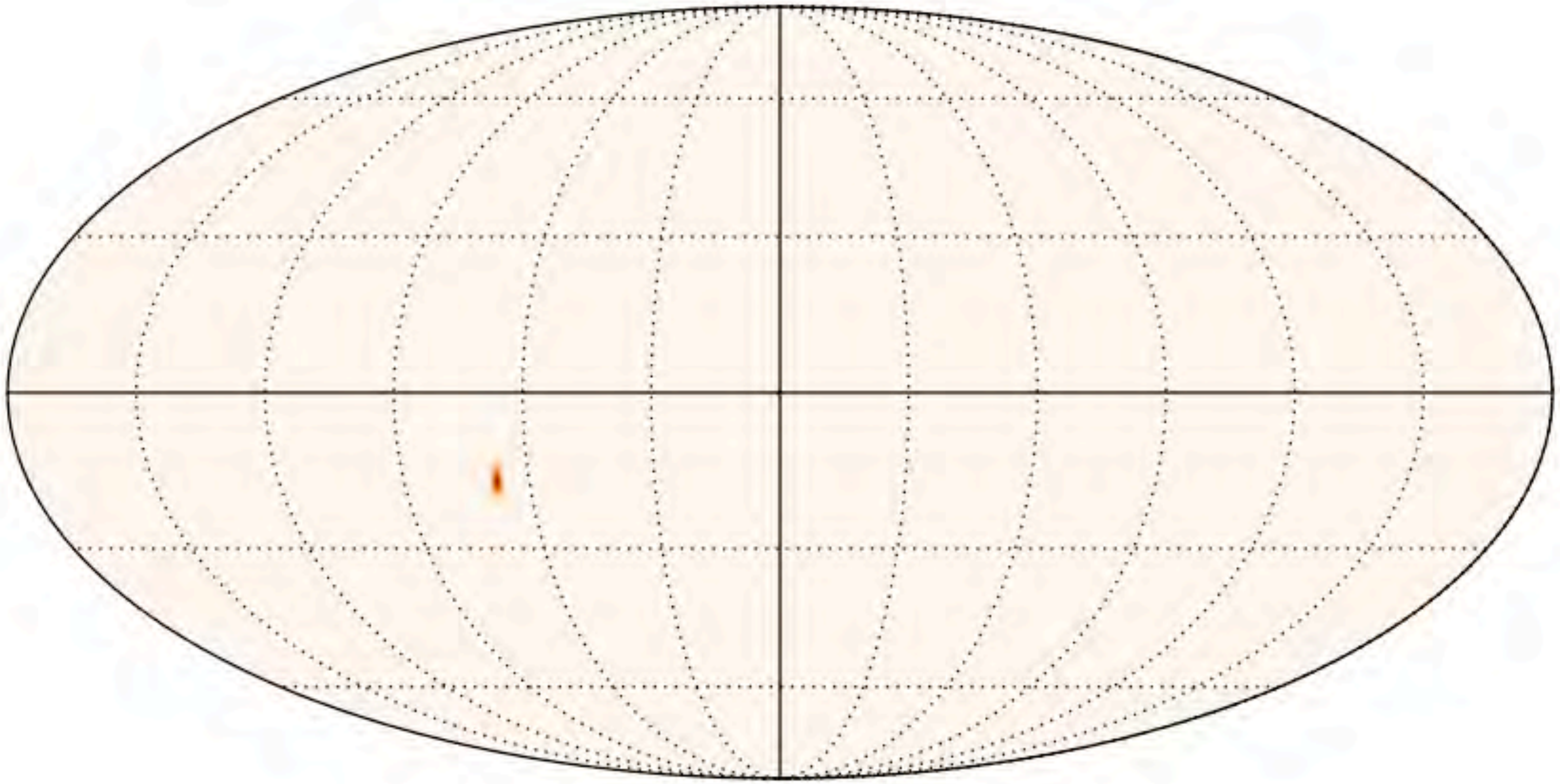
Hanford

# Triangulating the Source



Hanford + Livingston

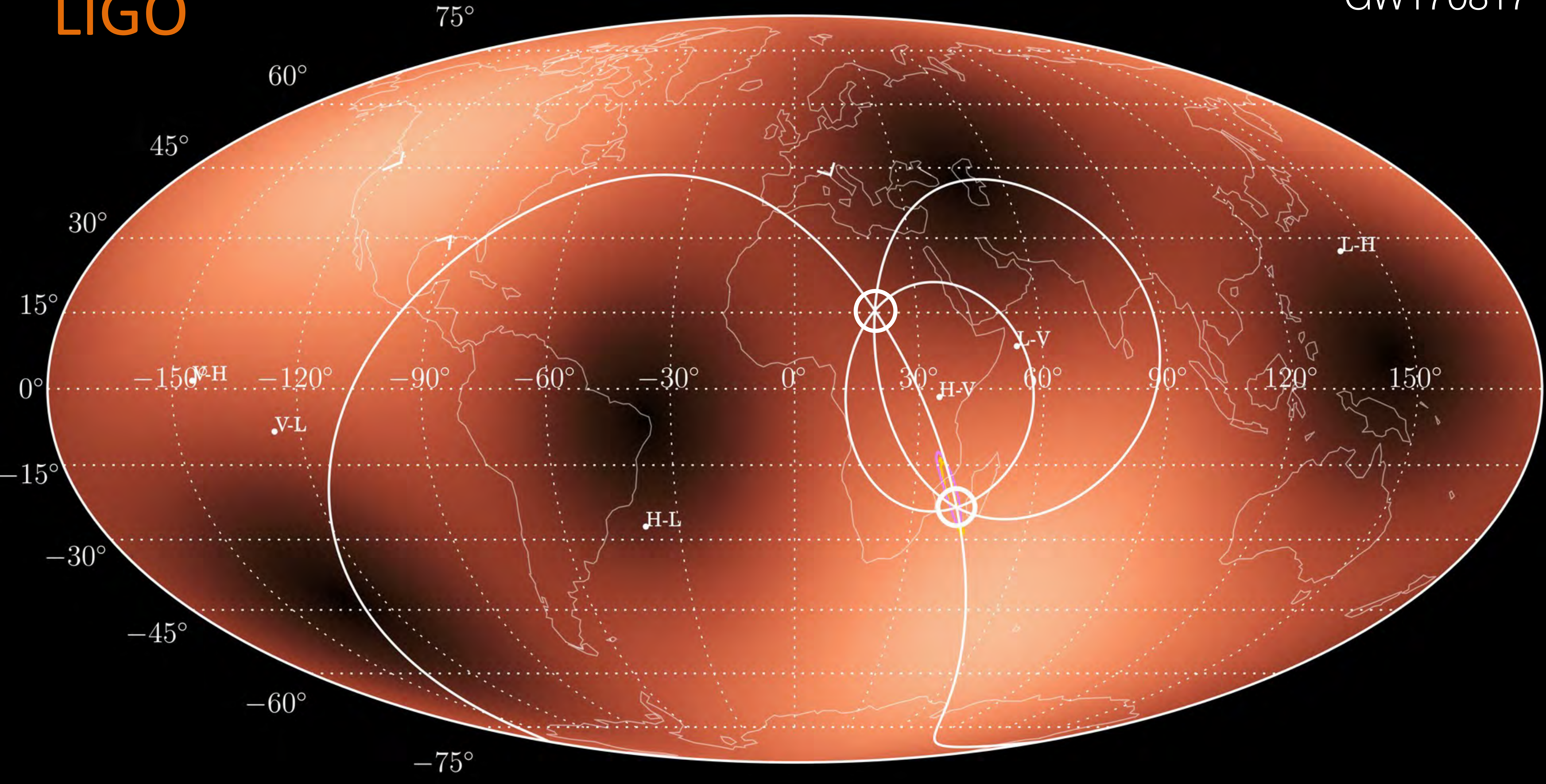
# Triangulating the Source



Hanford + Livingston + Virgo

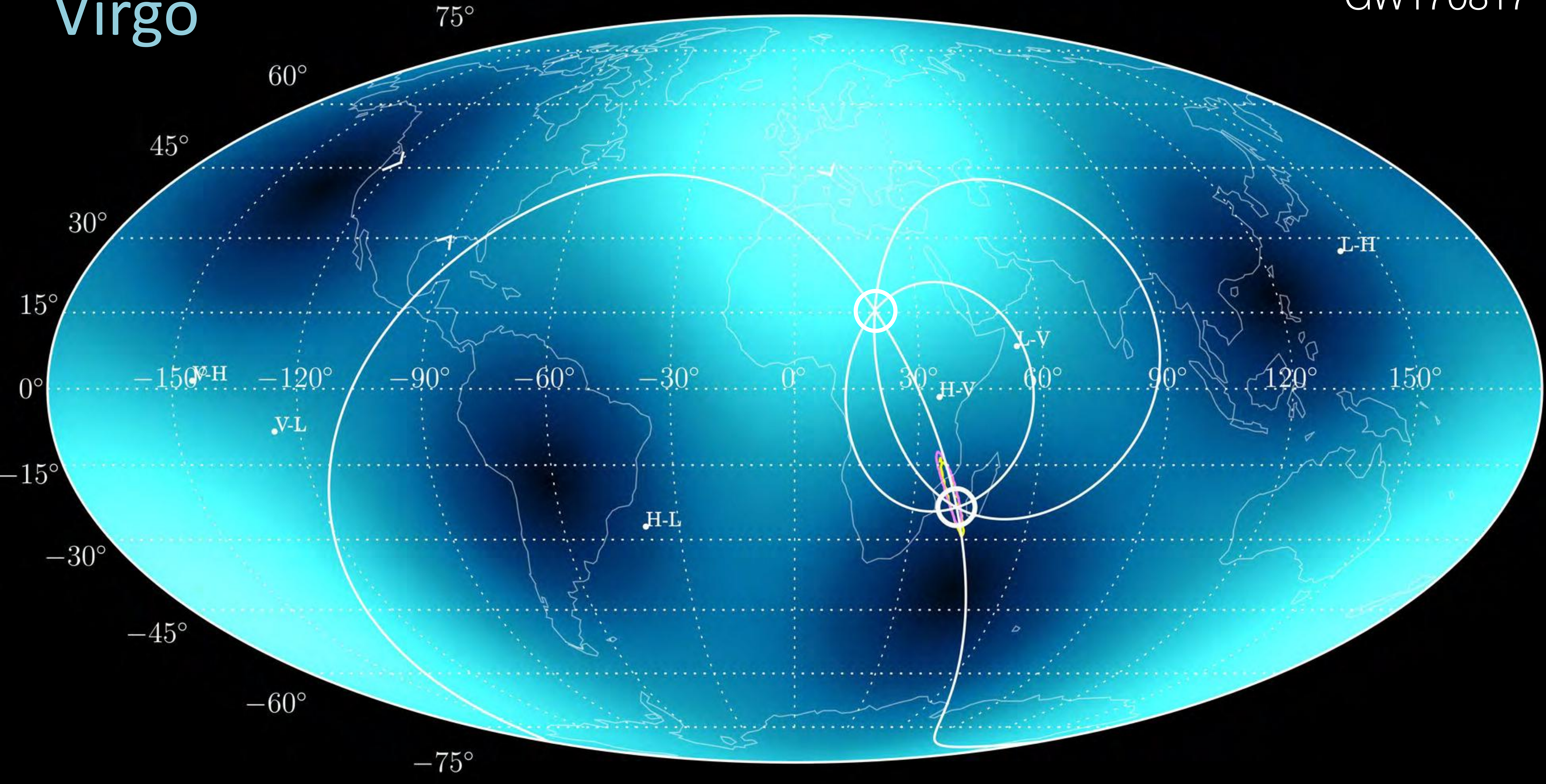
LIGO

GW170817

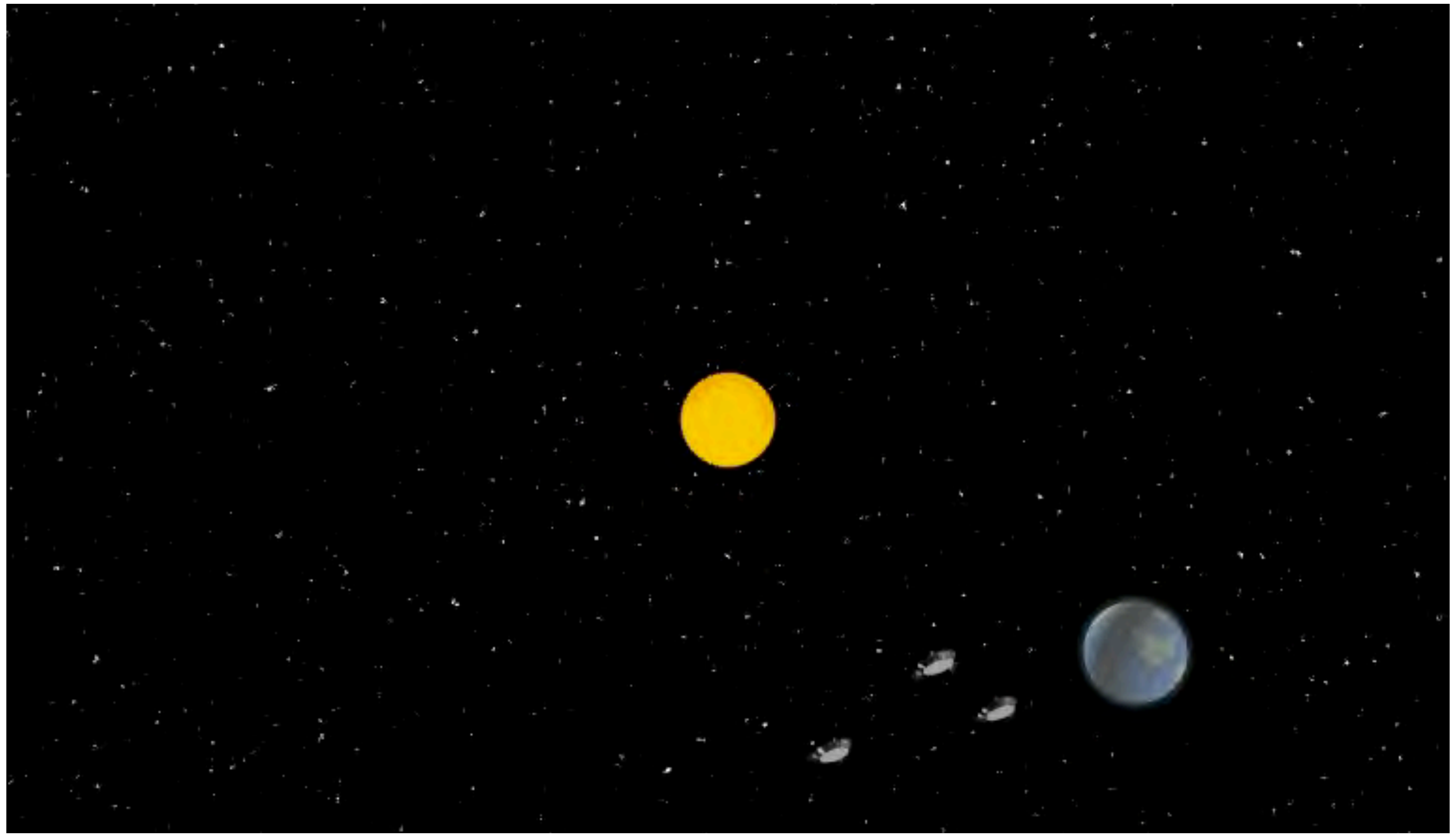


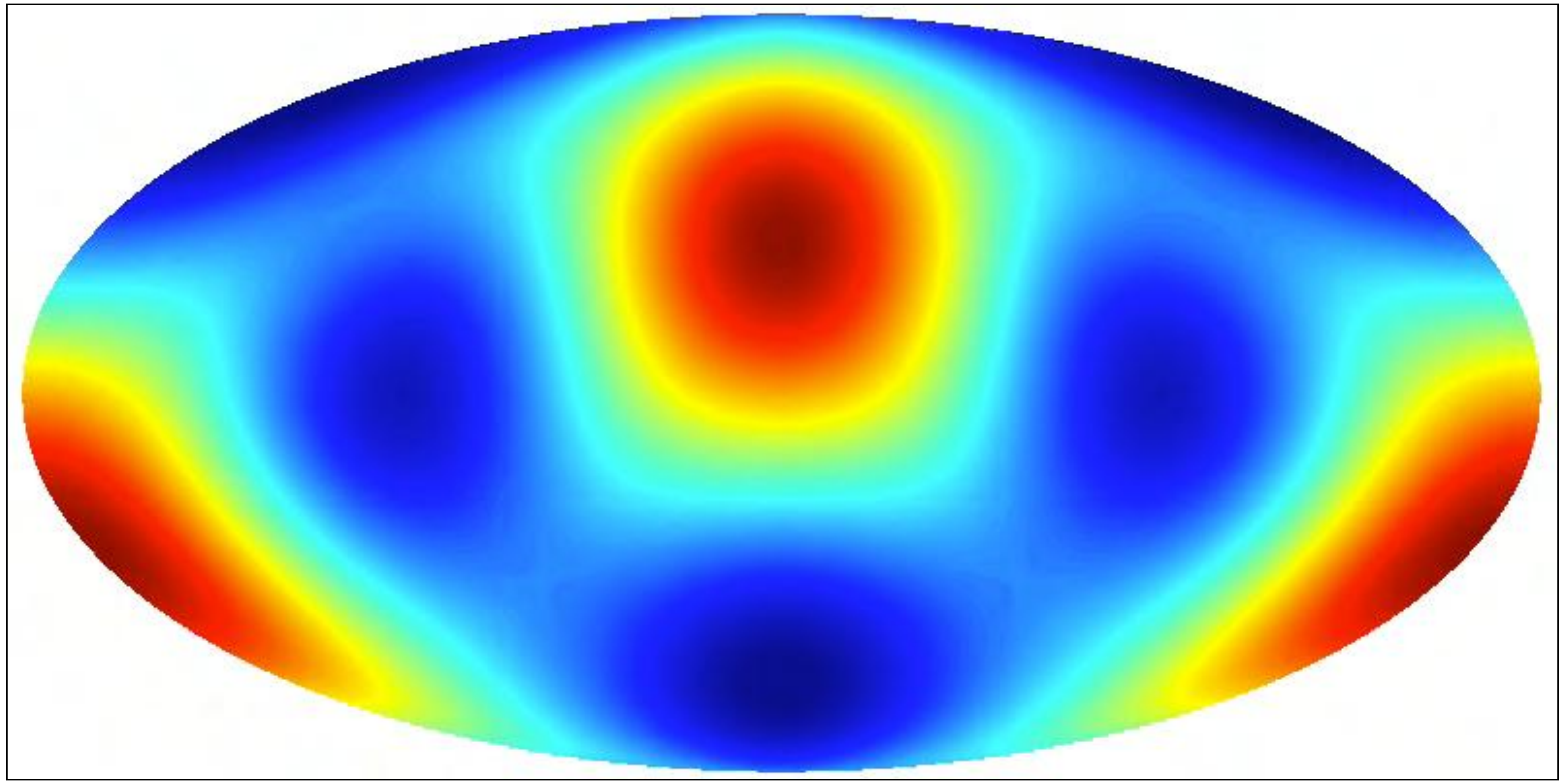
# Virgo

GW170817



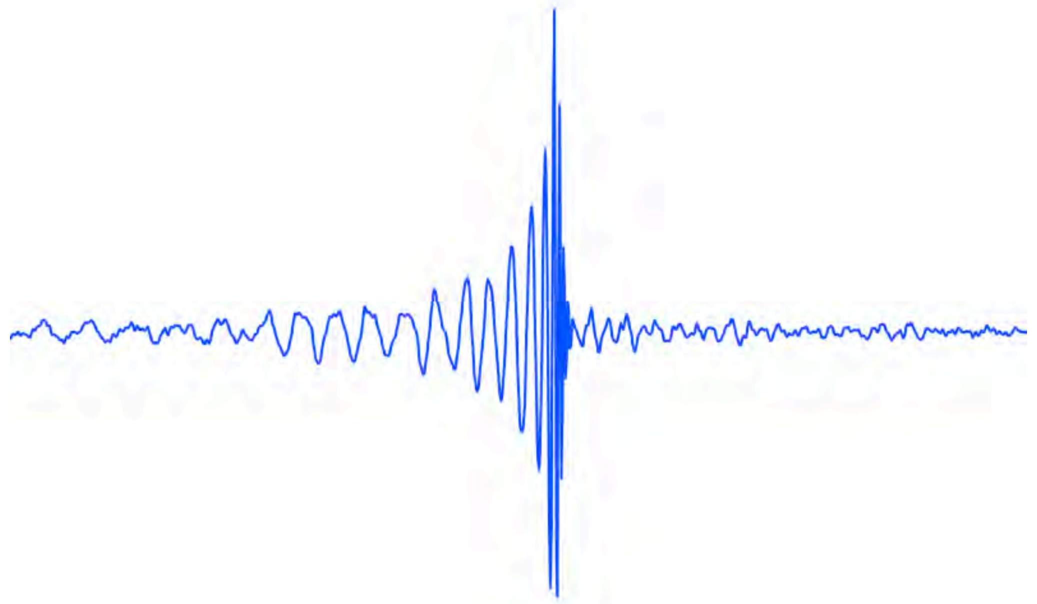
# Laser Interferometer Space Antenna

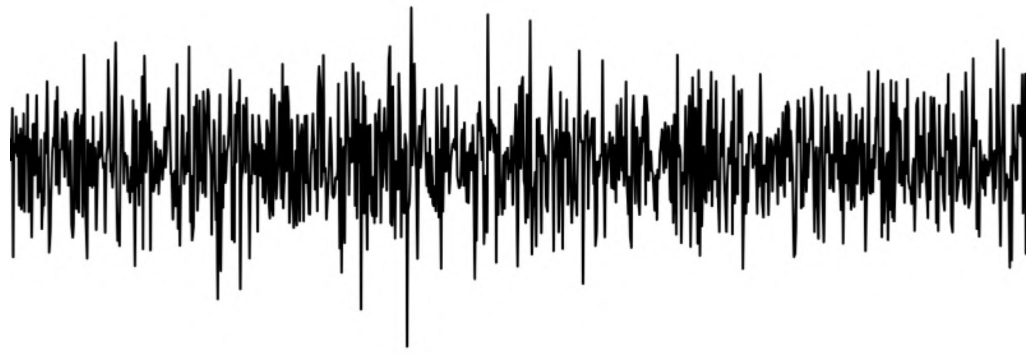




Low frequency response

# Data Analysis 101





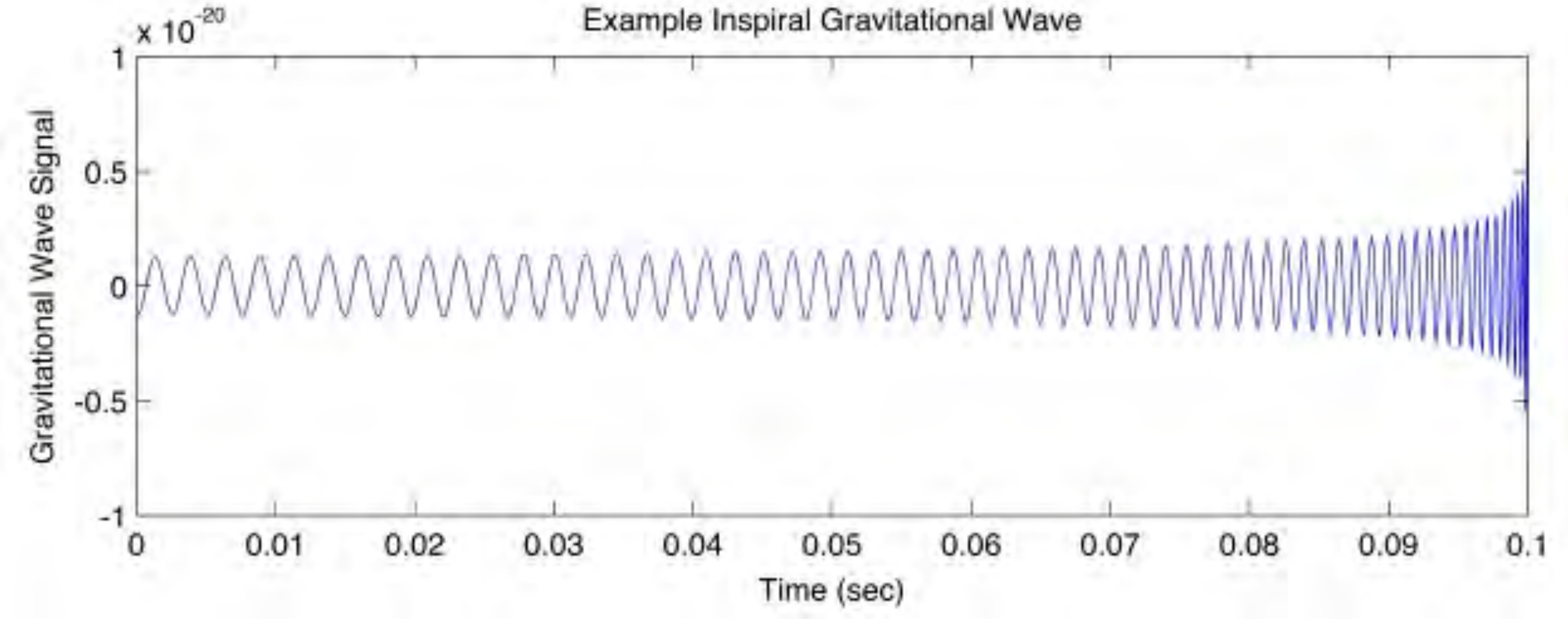
data - signal = noise

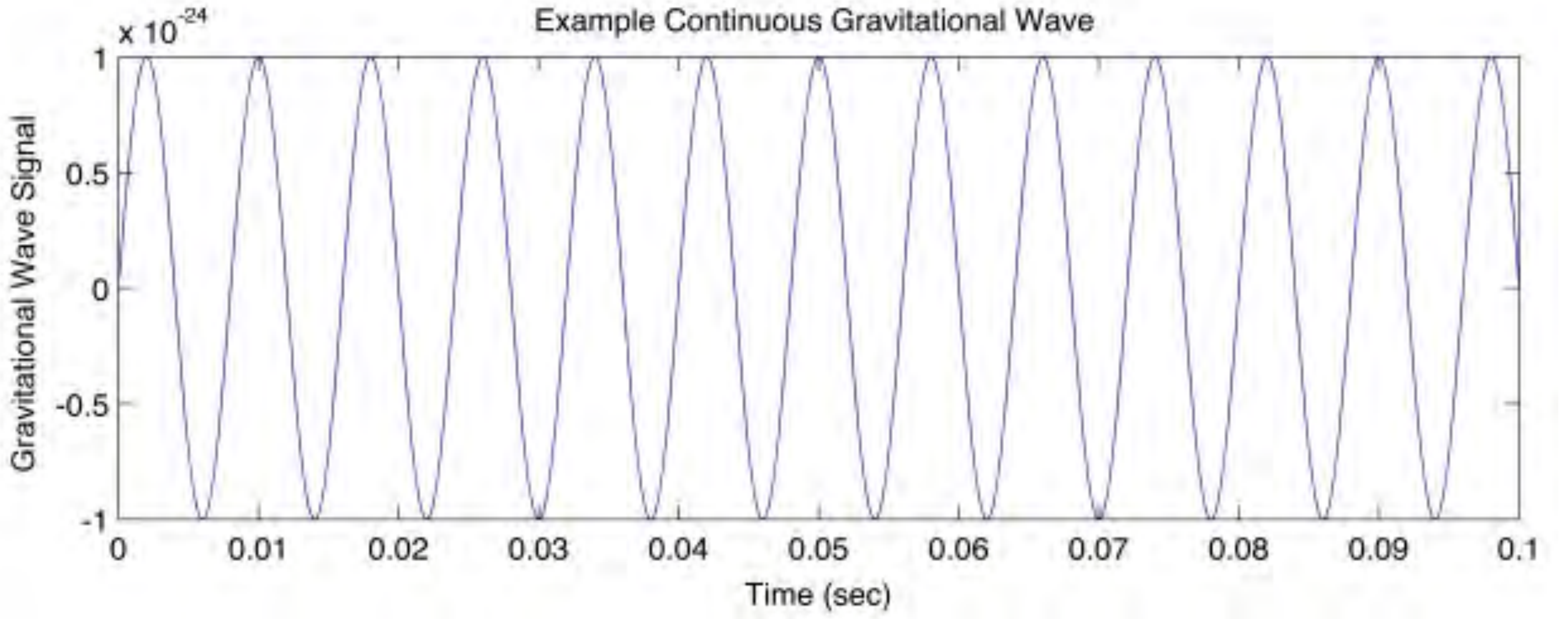
$$p(d|h) = p(d-h) = \frac{1}{(\det(2\pi\mathbf{C}))^{N/2}} e^{-\frac{1}{2}(d_i - h_i)C_{ij}^{-1}(d_j - h_j)}$$

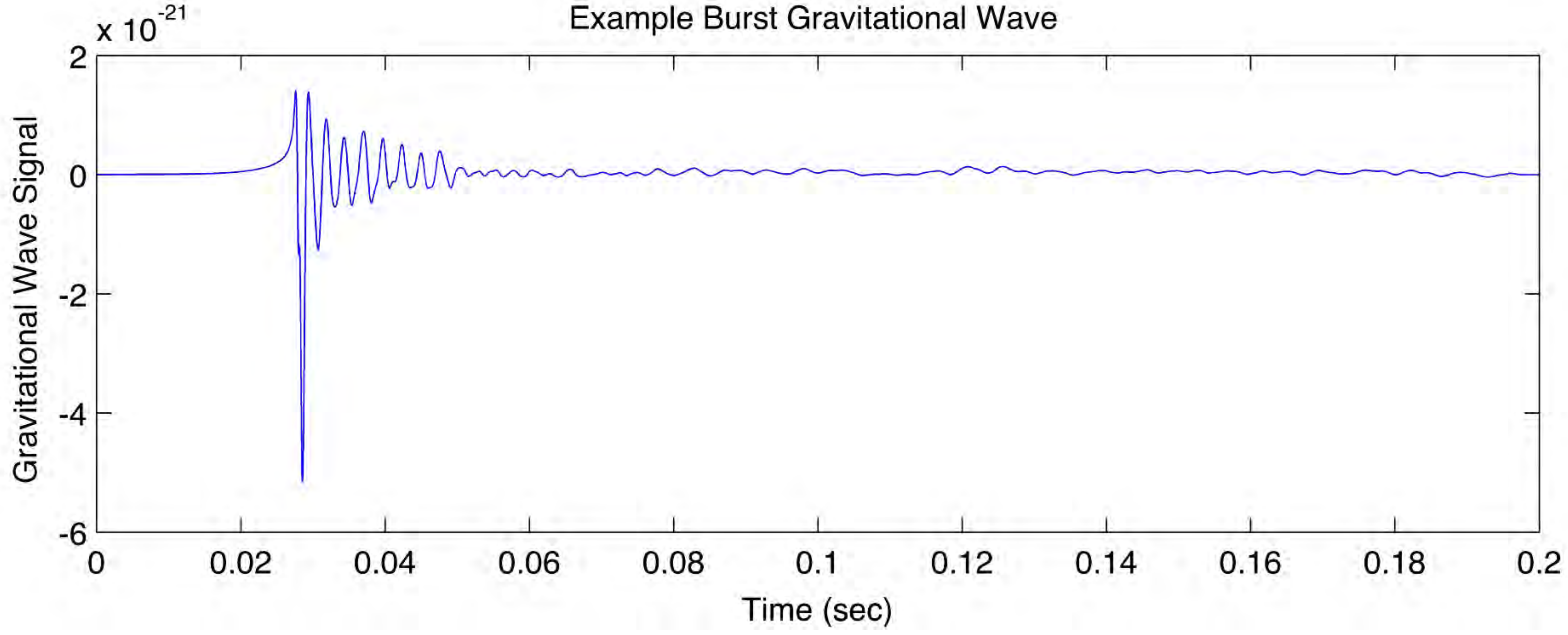
Likelihood

# Gravitational wave signal types = priors on h

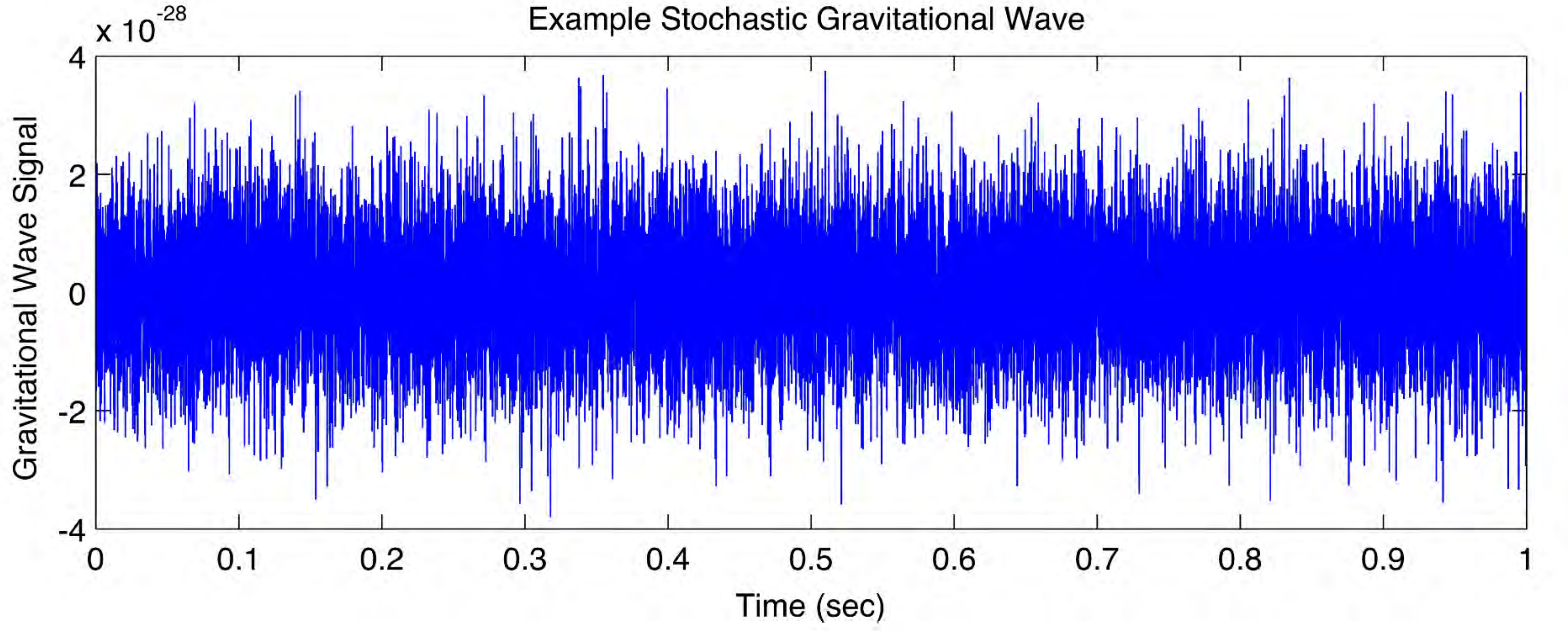
Well modeled - e.g. binary inspiral and merger




Poorly modeled - e.g. core collapse supernovae




Stochastic- e.g. phase transition in early universe

# Bayesian Inference

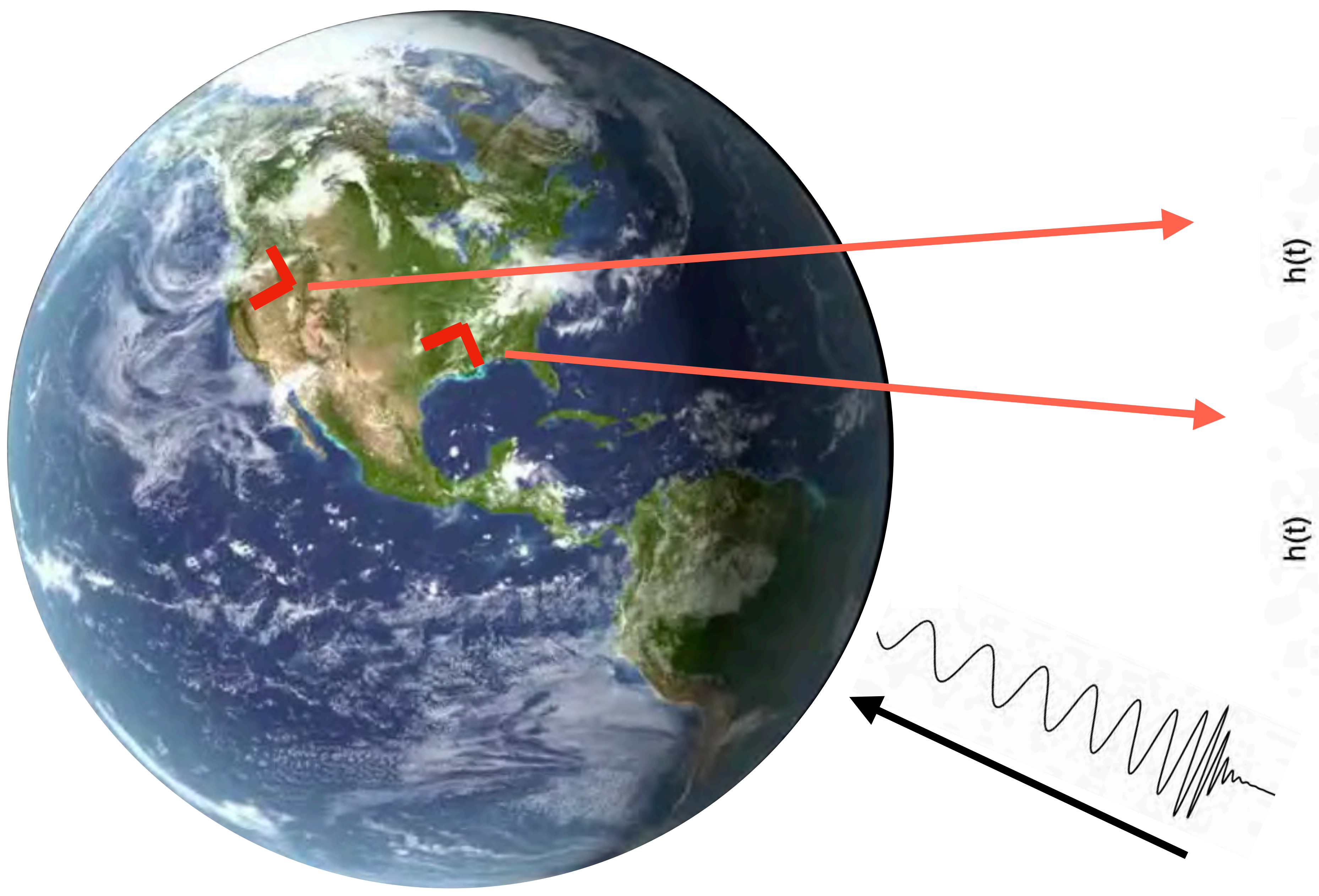
Prior
$p(h|M)$

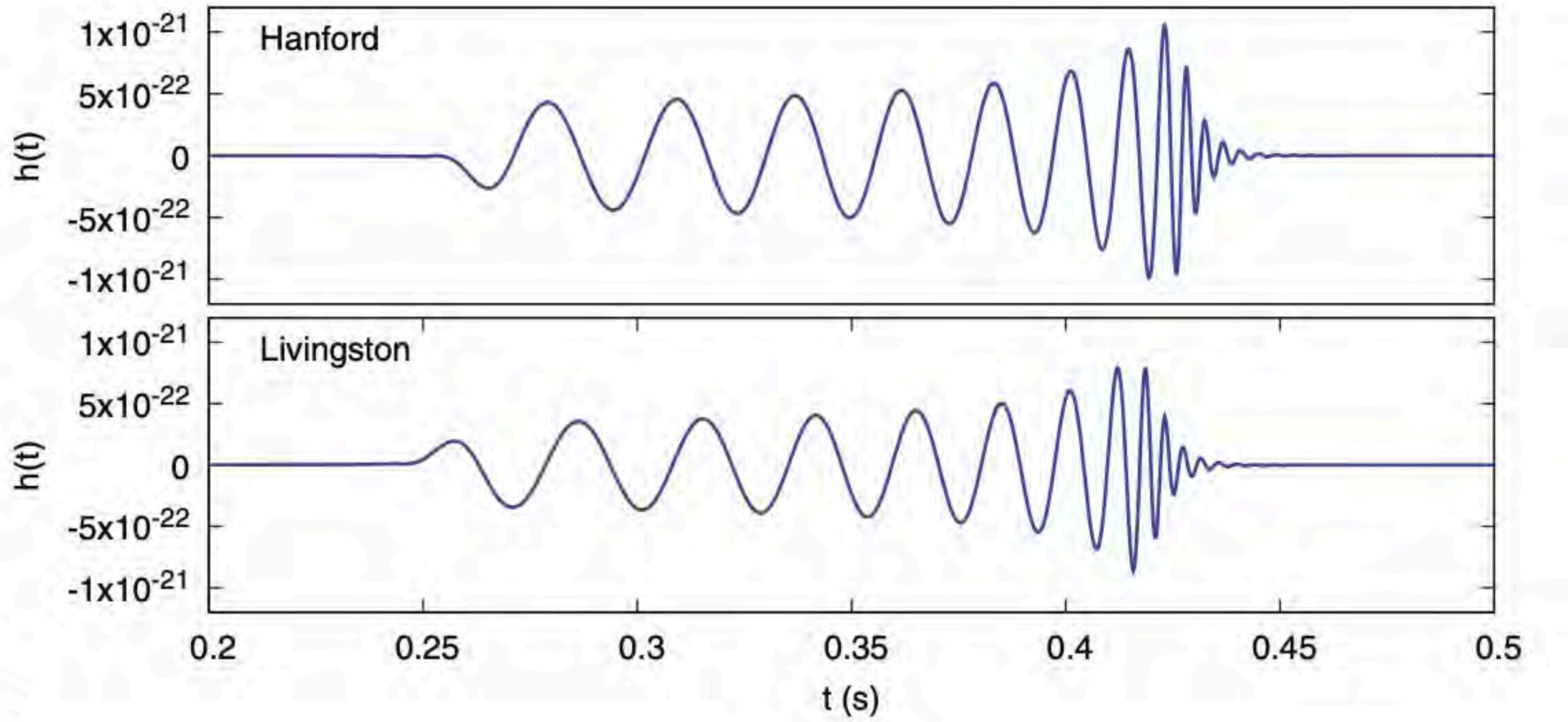
Likelihood
$p(d|h)$


$p(d|M)$
Evidence

$p(h|d, M)$
Posterior

# GW150914

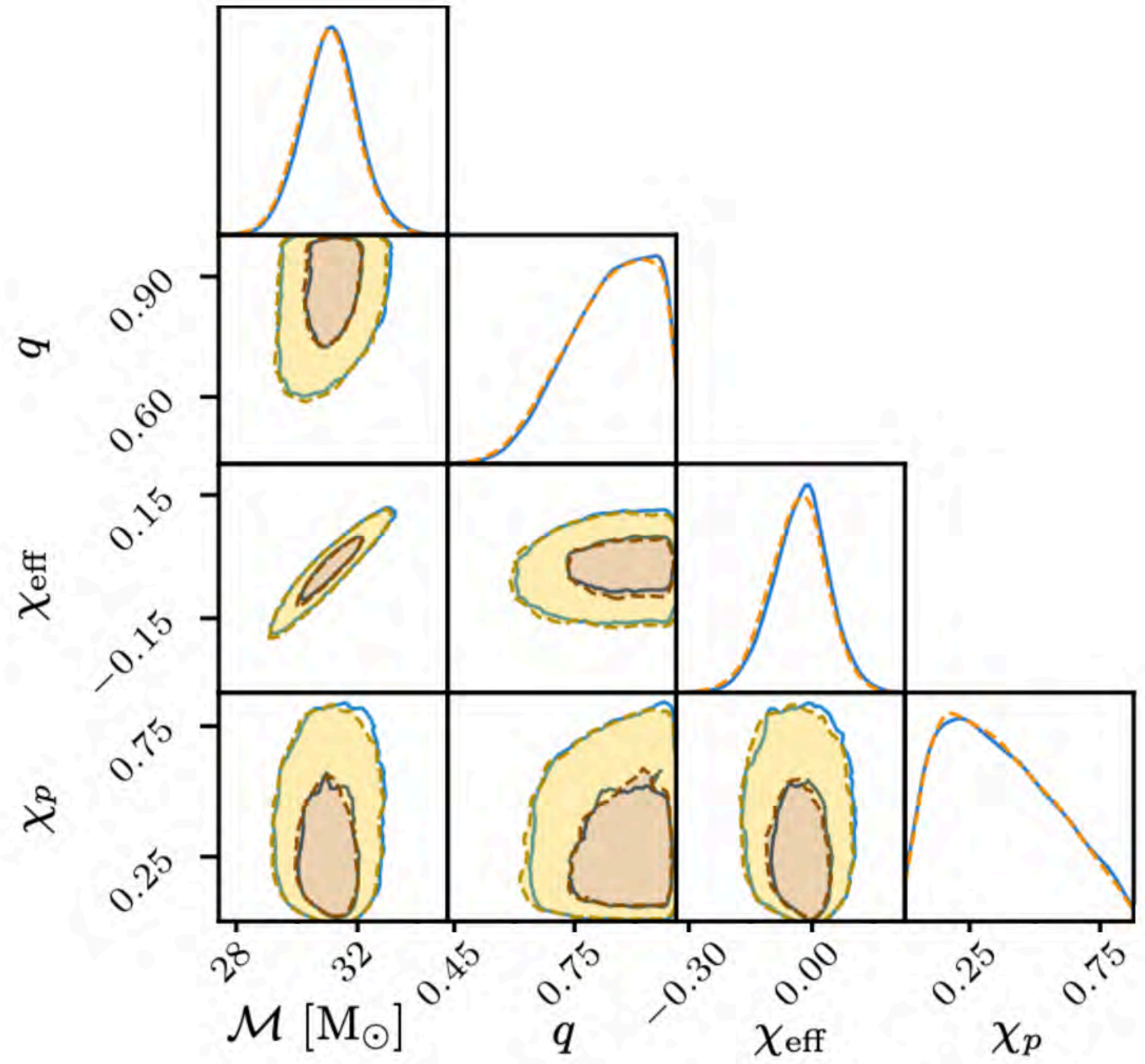


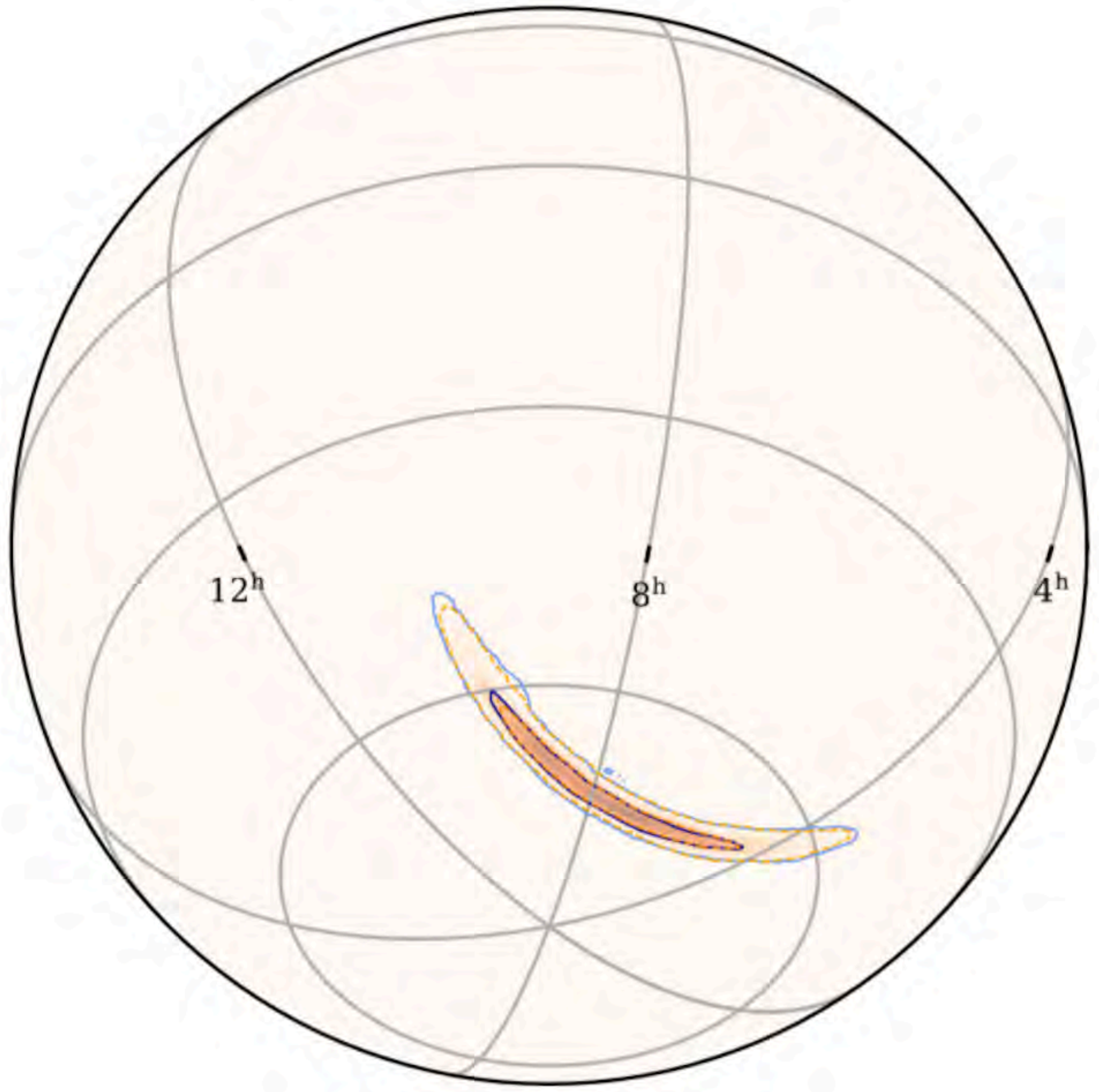


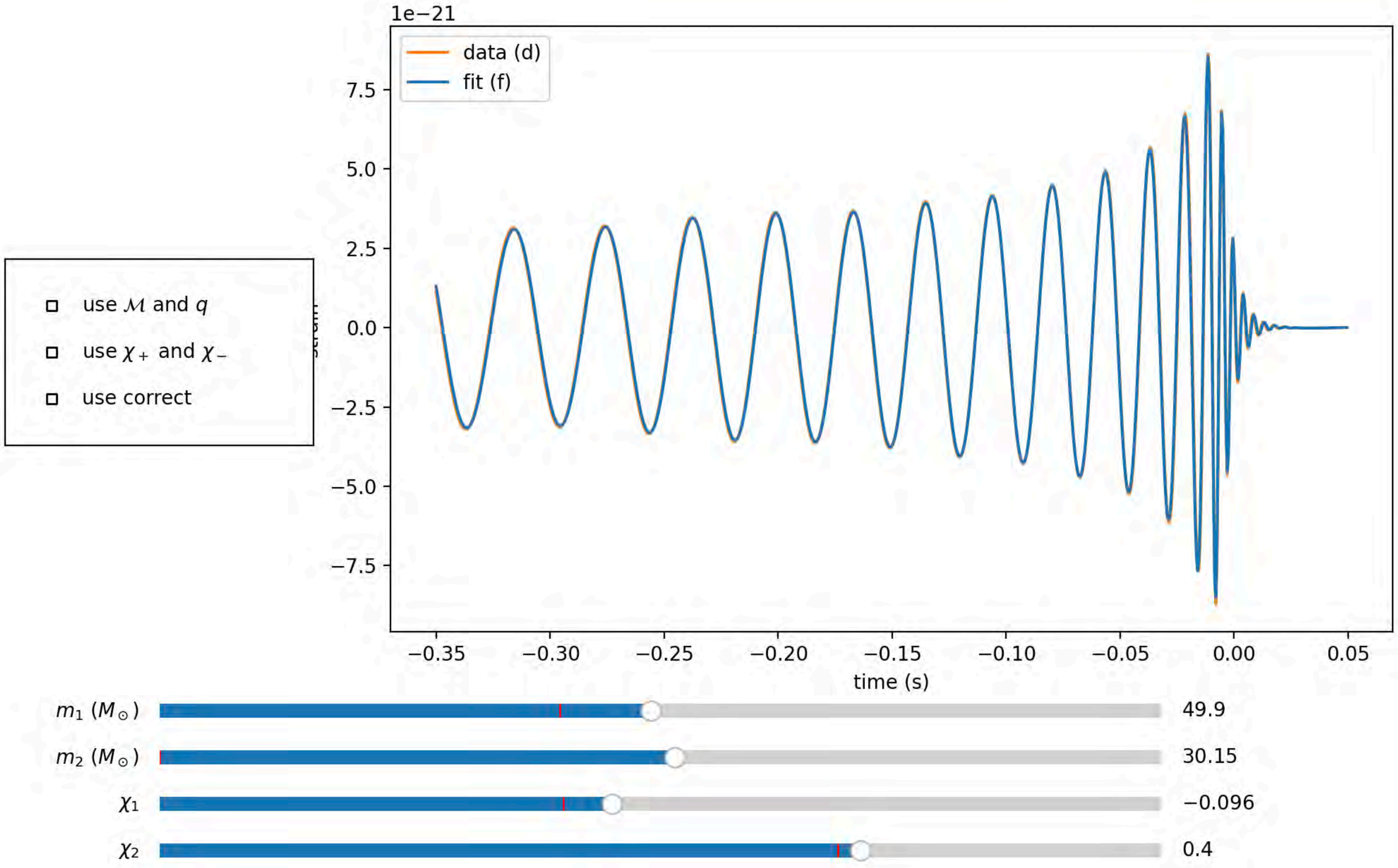

H-L Time delay 7 ms

H-L Phase Shift 2.9 radians

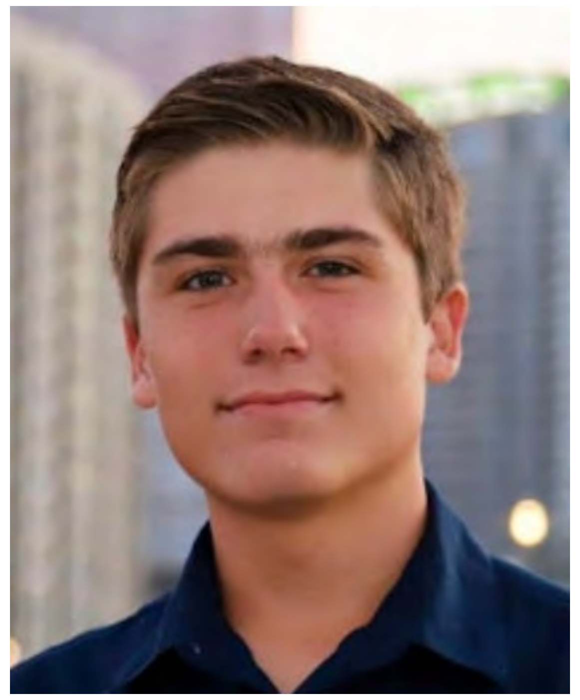
H-L Amplitude ratio 1.24



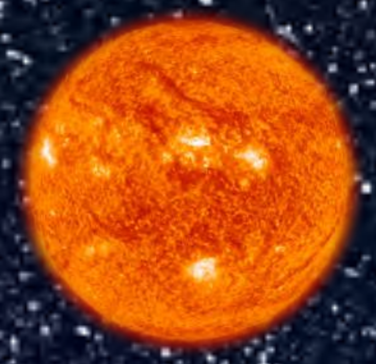



**Piper Morris**



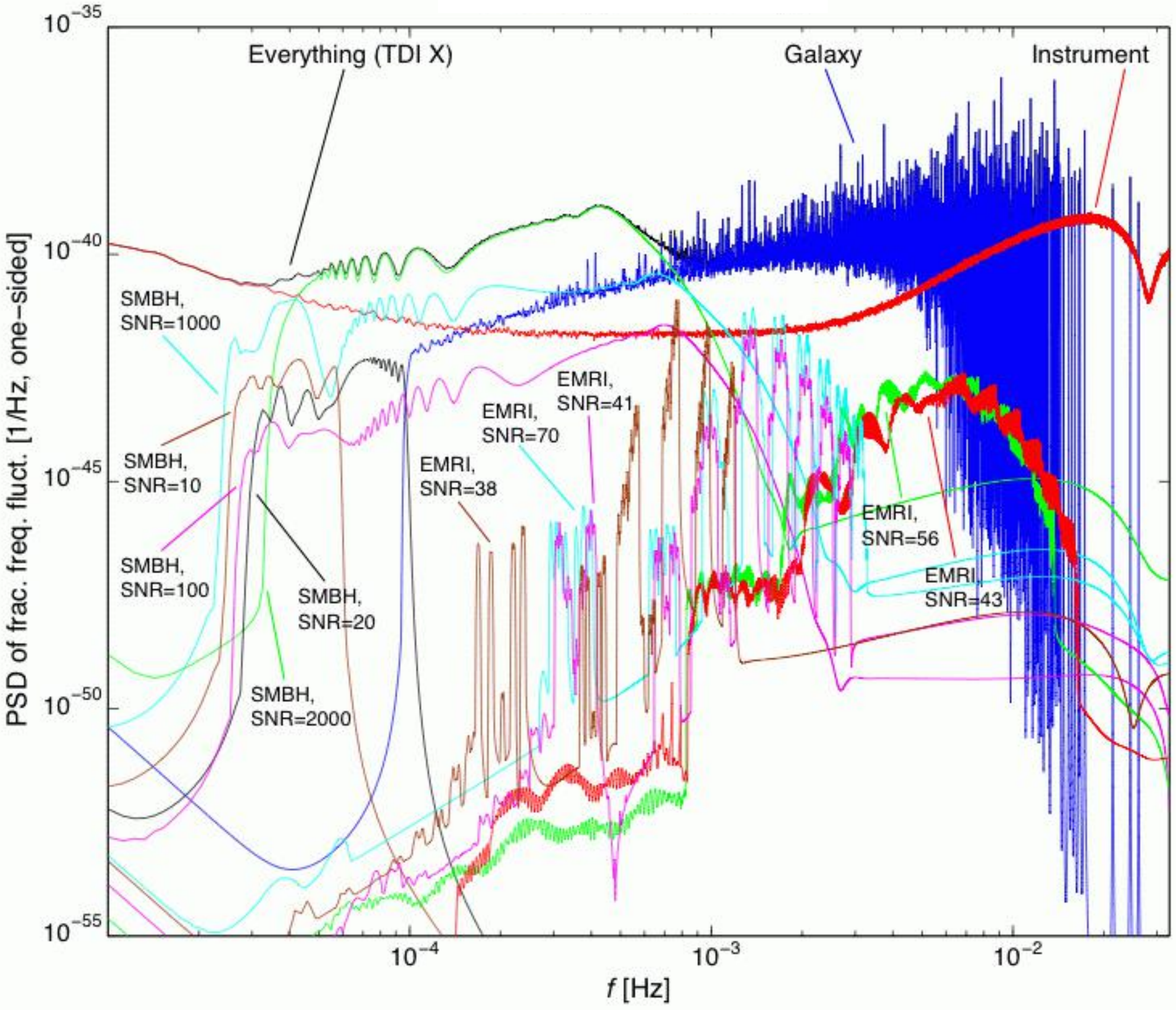
**Aiden Gunsderson**

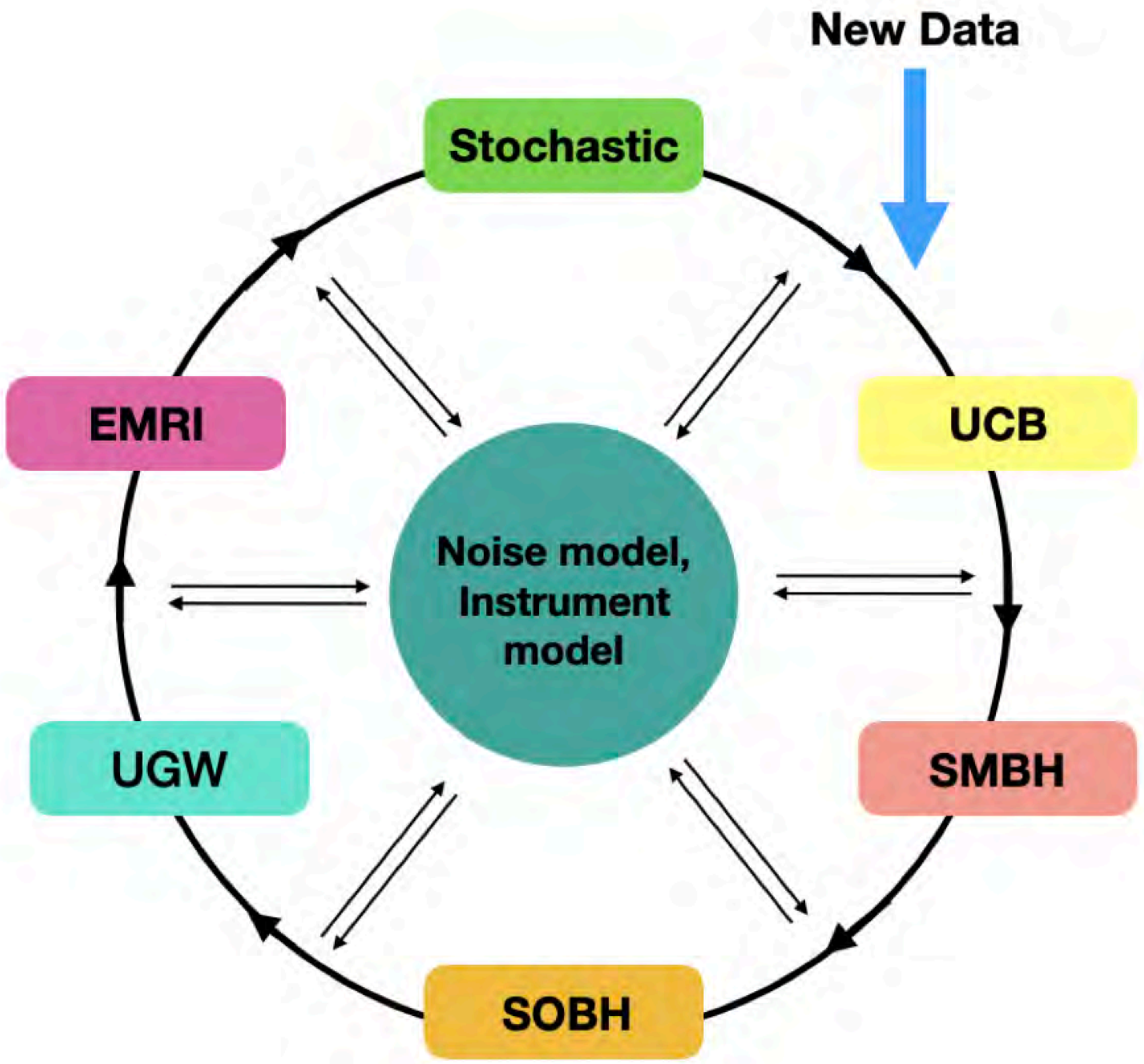
# LISA is not LIGO in Space



- Millions of overlapping signals
- Unknown number of detectable sources
- Non-stationary and non-Gaussian noise
- Data gaps and disturbances
- Time varying instrument response
- Complex signals, multiple harmonics

# LISA Global Fit - Simultaneously fitting tens of thousands of signals and noise




~200,000 parameter fit