From qubits to qutrits: entanglement of astrophysical neutrinos

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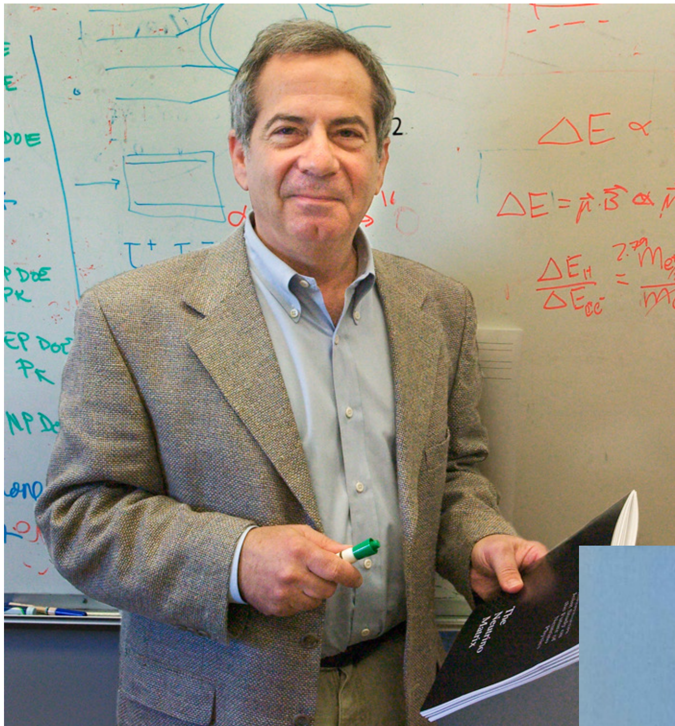
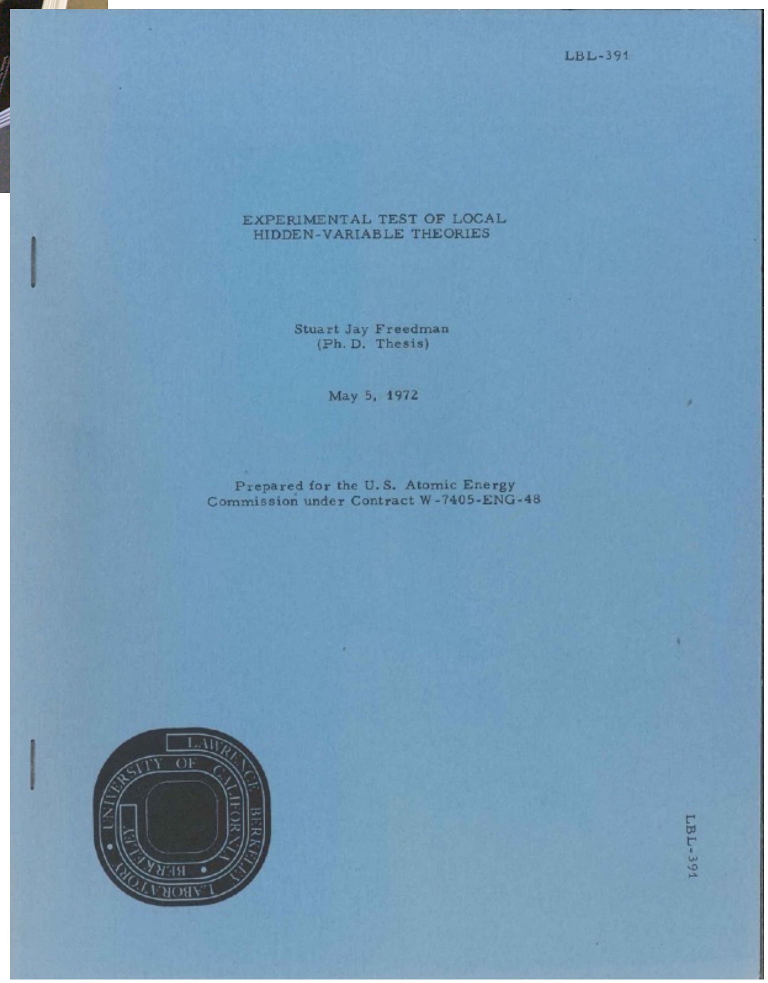
N3AS and RIKEN/iTHEMS joint meeting　N3ASと理研/iTHEMS合同会議

Network for Neutrinos, Nuclear Astrophysics, and Symmetries
PHYSICS FRONTIER CENTER



One of the earliest experiments on entanglement (Nobel Prize 2022 to Clauser) was carried out by Stuart Freedman who also started joint U.S.-Japan meetings in Hawaii.

A Tribute to Stuart Freedman



LBL-391

EXPERIMENTAL TEST OF LOCAL HIDDEN-VARIABLE THEORIES

Stuart Jay Freedman
(Ph. D. Thesis)

May 5, 1972

Prepared for the U. S. Atomic Energy Commission under Contract W-7405-ENG-48

LAWRENCE BERKELEY LABORATORY UNIVERSITY OF CALIFORNIA BERKELEY

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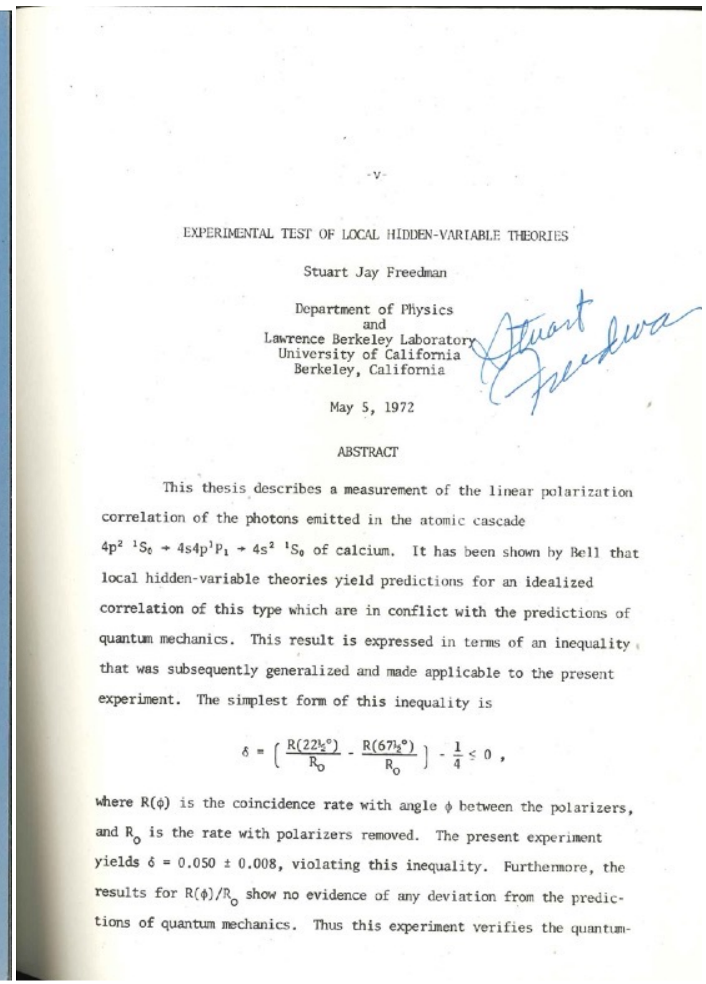

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Stuart Jay Freedman

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and
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Berkeley, California

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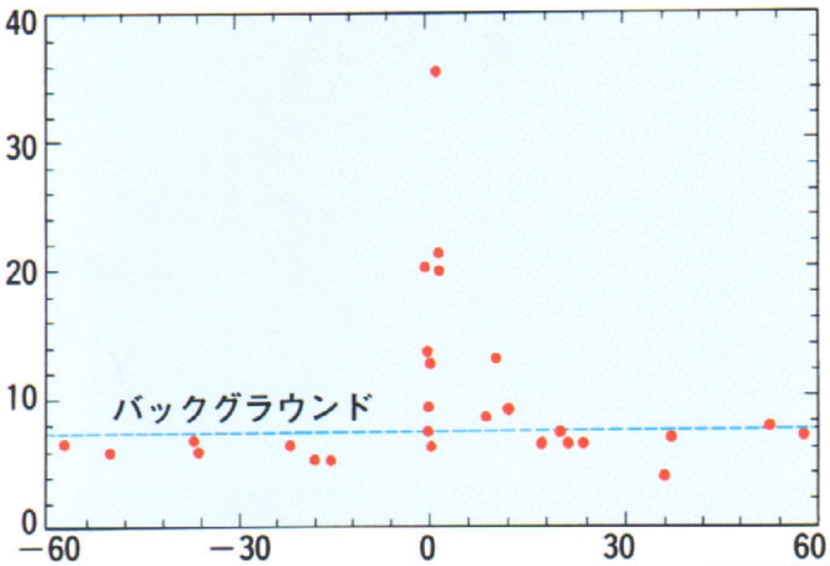
ABSTRACT

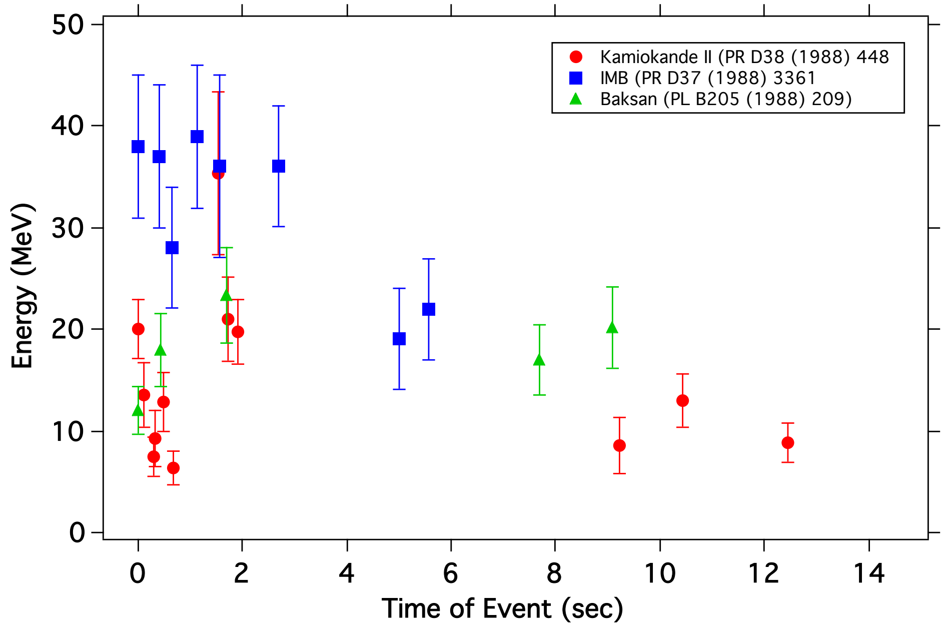
This thesis describes a measurement of the linear polarization correlation of the photons emitted in the atomic cascade $4p^2\ {}^1S_0 \rightarrow 4s4p\ {}^1P_1 \rightarrow 4s^2\ {}^1S_0$ of calcium. It has been shown by Bell that local hidden-variable theories yield predictions for an idealized correlation of this type which are in conflict with the predictions of quantum mechanics. This result is expressed in terms of an inequality that was subsequently generalized and made applicable to the present experiment. The simplest form of this inequality is

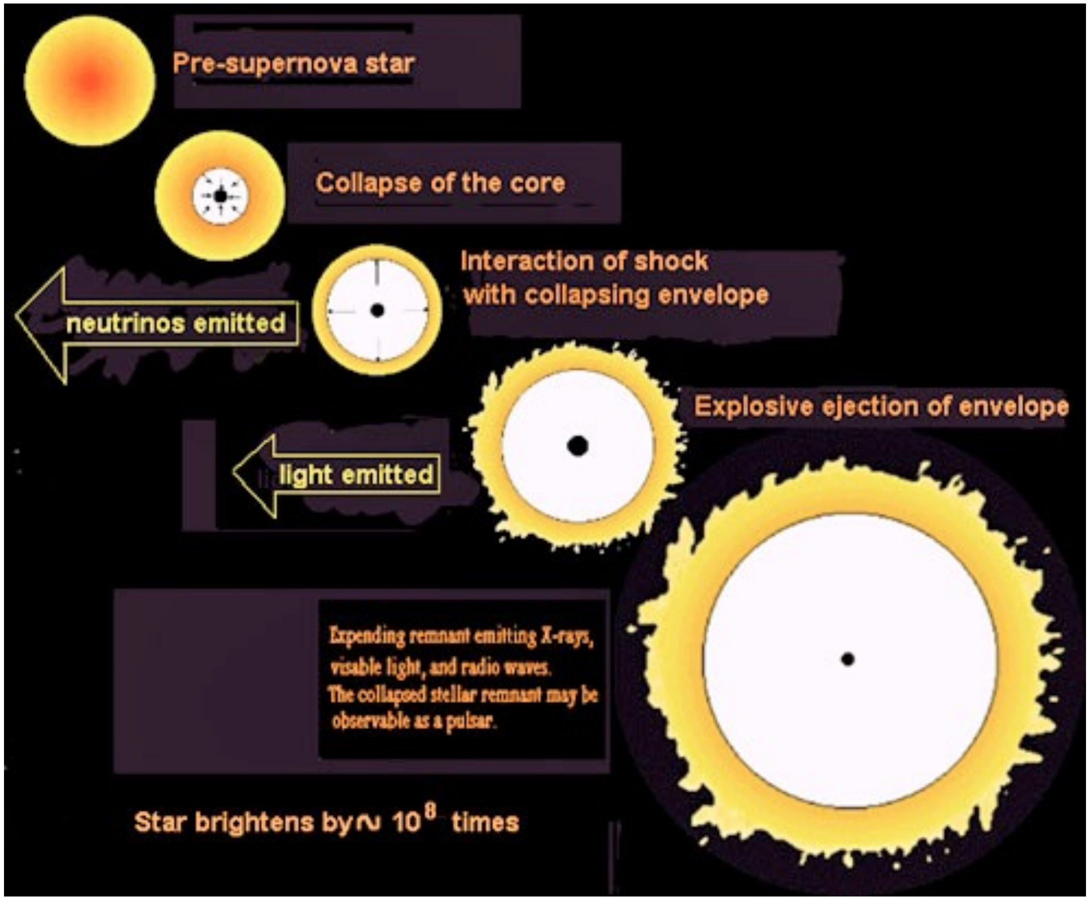
$$\delta = \left[\frac{R(22\tfrac{1}{2}°)}{R_0} - \frac{R(67\tfrac{1}{2}°)}{R_0} \right] - \frac{1}{4} \leq 0 \ ,$$

where $R(\phi)$ is the coincidence rate with angle ϕ between the polarizers, and R_0 is the rate with polarizers removed. The present experiment yields $\delta = 0.050 \pm 0.008$, violating this inequality. Furthermore, the results for $R(\phi)/R_0$ show no evidence of any deviation from the predictions of quantum mechanics. Thus this experiment verifies the quantum-

Neutrinos from core-collapse supernovae 1987A



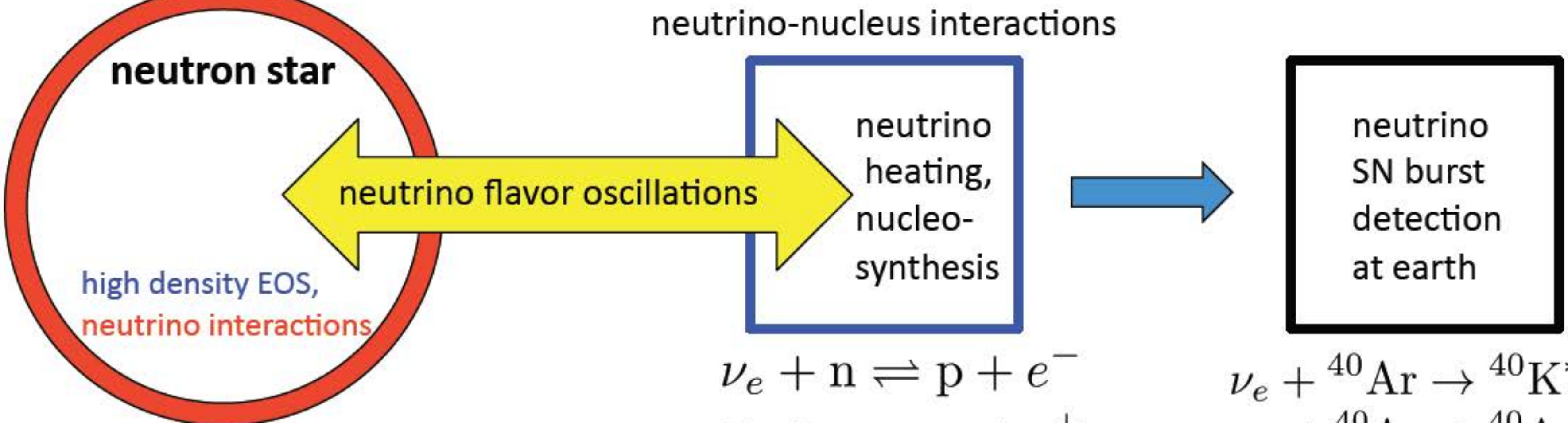




- $M_{prog} \geq 8\ M_{sun} \Rightarrow \Delta E \approx 10^{53}$ ergs $\approx 10^{59}$ MeV
- 99% of the energy is carried away by neutrinos and antineutrinos with $10 \leq E_\nu \leq 30$ MeV $\Rightarrow 10^{58}$ neutrinos

Understanding a core-collapse supernova requires answers to a variety of questions some of which need to be answered, both theoretically and experimentally.

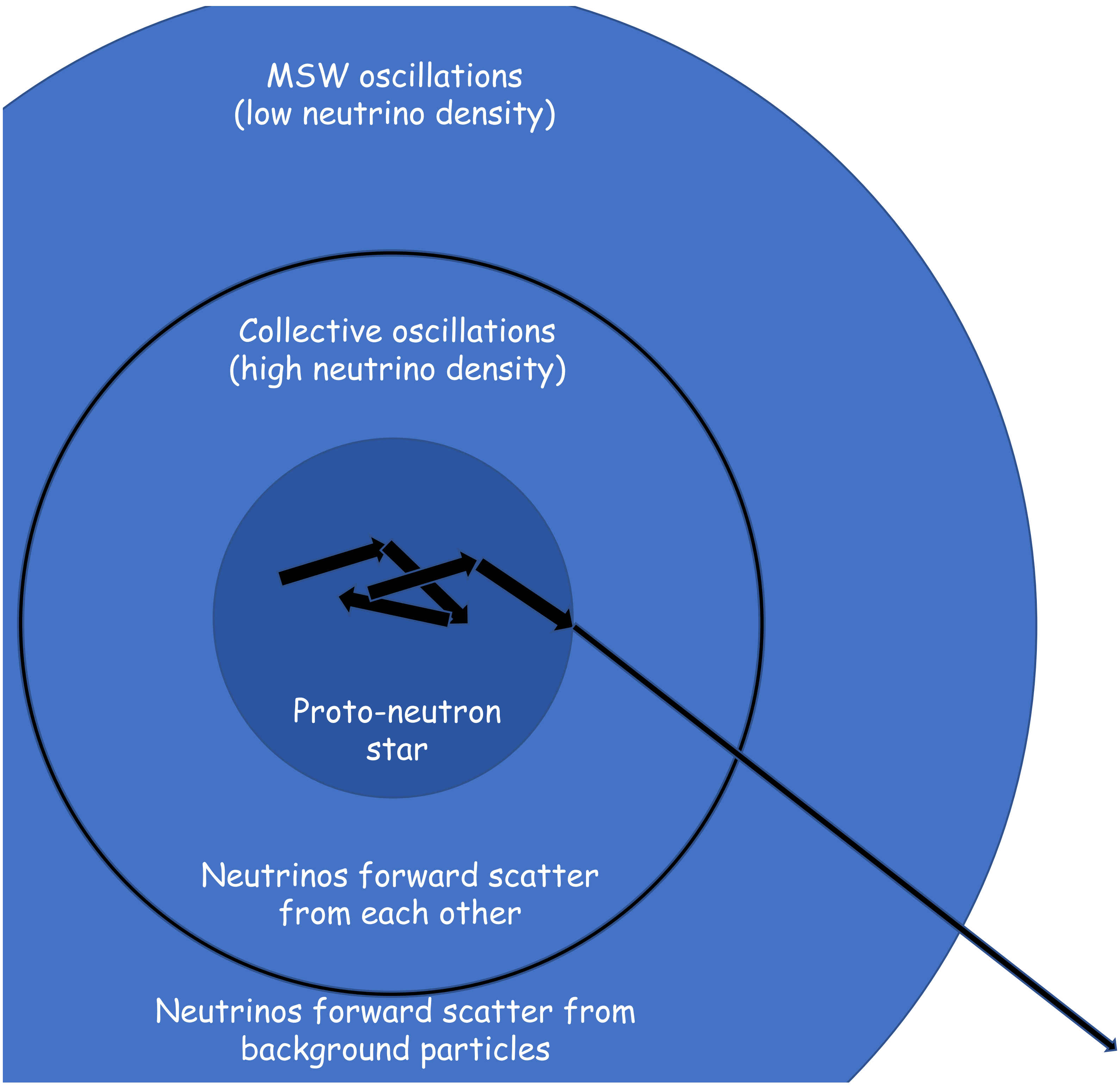
Balantekin and Fuller, Prog. Part. Nucl. Phys. **71** 162 (2013)

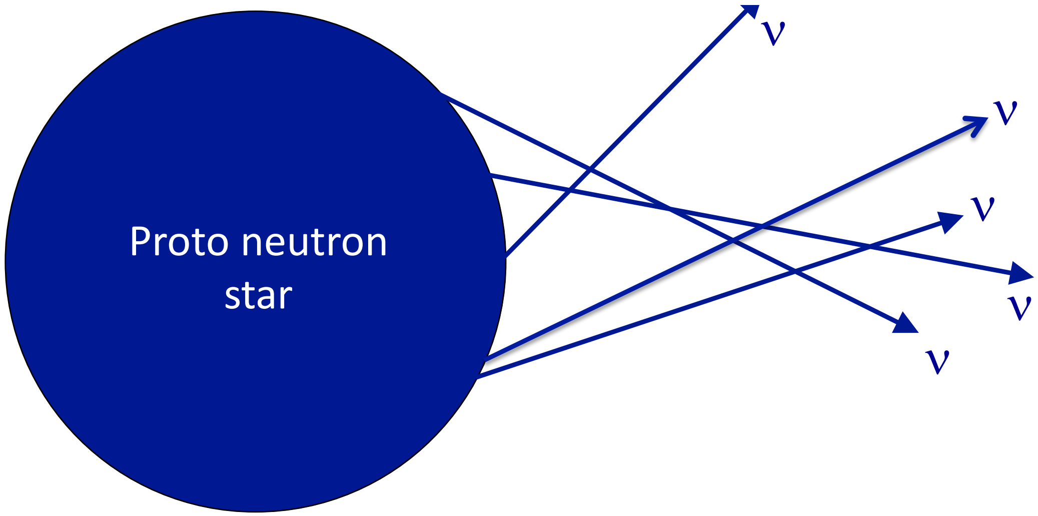


$$\nu_e + \mathrm{n} \rightleftharpoons \mathrm{p} + e^-$$
$$\bar{\nu}_e + \mathrm{p} \rightleftharpoons \mathrm{n} + e^+$$

$$\nu_e + {}^{40}\mathrm{Ar} \to {}^{40}\mathrm{K}^* + e^-$$
$$\nu_x + {}^{40}\mathrm{Ar} \to {}^{40}\mathrm{Ar}^* + \nu_x$$

Neutron-to-proton ratio depends on relative intensities of electron neutrinos and electron antineutrinos, which in turn depend on neutrino oscillations


MSW oscillations
(low neutrino density)
Collective oscillations
(high neutrino density)
Proto-neutron
star
Neutrinos forward scatter
from each other
Neutrinos forward scatter from
background particles



Energy released in a core-collapse SN: $\Delta E \approx 10^{53}$ ergs $\approx 10^{59}$ MeV
99% of this energy is carried away by neutrinos and antineutrinos!
~ 10^{58} Neutrinos!
This necessitates including the effects of νν interactions!

$$H = \underbrace{\sum a^{\dagger} a}_{\text{ν oscillations MSW effect}} + \underbrace{\sum (1 - \cos\varphi) a^{\dagger} a^{\dagger} a a}_{\text{neutrino-neutrino interactions}}$$

The second term makes the physics of a neutrino gas in a core-collapse supernova a very interesting many-body problem, driven by weak interactions.

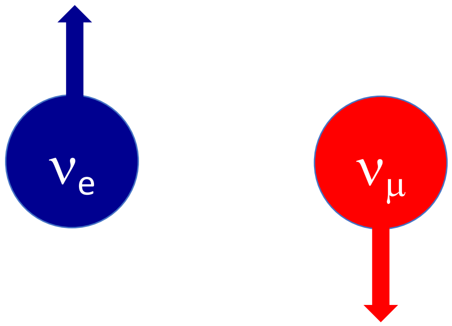
Neutrino-neutrino interactions lead to novel collective and emergent effects, such as conserved quantities and interesting features in the neutrino energy spectra (spectral "swaps" or "splits").

$$\frac{\partial\rho}{\partial t} = -i[H,\rho] + C(\rho)$$

H = neutrino mixing
+ forward scattering of neutrinos off other background particles (MSW)
+ forward scattering of neutrinos off each other

C = collisions

Neutrino flavor isospin



$$\hat{J}_+ = a_e^\dagger a_\mu \qquad \hat{J}_- = a_\mu^\dagger a_e$$

$$\hat{J}_0 = \frac{1}{2}\left(a_e^\dagger a_e - a_\mu^\dagger a_\mu\right)$$

These operators can be written in either mass or flavor basis

Free neutrinos (only mixing)

$$\hat{H} = \frac{m_1^2}{2E} a_1^\dagger a_1 + \frac{m_2^2}{2E} a_2^\dagger a_2 + (\cdot\cdot)\hat{1}$$

$$= \frac{\delta m^2}{4E}\cos 2\theta\left(-2\hat{J}_0\right) + \frac{\delta m^2}{4E}\sin 2\theta\left(\hat{J}_+ + \hat{J}_-\right) + (\cdot\cdot)'\,\hat{1}$$

Interacting with background electrons

$$\hat{H} = \left[\frac{\delta m^2}{4E}\cos 2\theta - \frac{1}{\sqrt{2}} G_F N_e\right]\left(-2\hat{J}_0\right) + \frac{\delta m^2}{4E}\sin 2\theta\left(\hat{J}_+ + \hat{J}_-\right) + (\cdot\cdot)''\,\hat{1}$$

Note that

$$J_o = \frac{1}{2}\left(a_e^\dagger a_e - a_\mu^\dagger a_\mu\right)$$

$$N = \left(a_e^\dagger a_e + a_\mu^\dagger a_\mu\right) = \text{ constant}$$

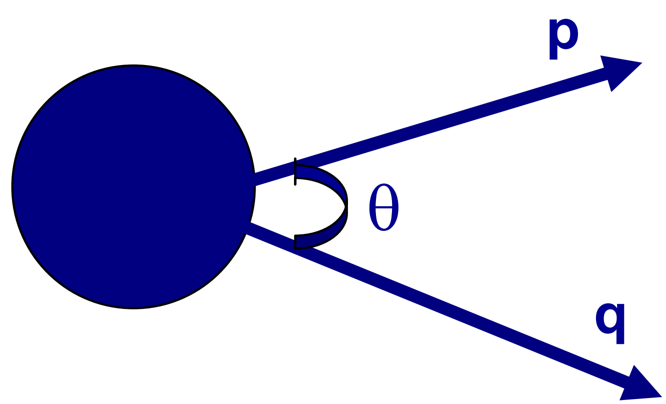
Hence $\sum P_0 \equiv \text{ Tr }(\rho J_0)$ is an observable giving numbers of neutrinos of each flavor

Note $\boxed{\rho = \frac{1}{2}\left(1 + \vec{\sigma} \cdot \vec{P}\right)}$ single neutrino density matrix

Neutrino-Neutrino Interactions

Smirnov, Fuller, Qian, Pantaleone, Sawyer, McKellar, Friedland, Lunardini, Raffelt, Duan, Balantekin, Volpe, Kajino, Pehlivan ...

$$\hat{H}_{\nu\nu} = \frac{\sqrt{2}G_F}{V}\int dp\, dq \left(1-\cos\theta_{pq}\right)\vec{\mathbf{J}}_p \cdot \vec{\mathbf{J}}_q$$



This term makes the physics of a neutrino gas in a core-collapse supernova a genuine many-body problem

$$\hat{H} = \int dp \left(\frac{\delta m^2}{2E}\vec{\mathbf{B}}\cdot\vec{\mathbf{J}}_p - \sqrt{2}G_F N_e \mathbf{J}_p^0\right) + \frac{\sqrt{2}G_F}{V}\int dp\, dq\left(1-\cos\theta_{pq}\right)\vec{\mathbf{J}}_p\cdot\vec{\mathbf{J}}_q$$

$$\vec{\mathbf{B}} = \left(\sin 2\theta,\ 0, -\cos 2\theta\right)$$

Neutrino-neutrino interactions lead to novel collective and emergent effects, such as conserved quantities and interesting features in the neutrino energy spectra (spectral “swaps” or “splits”).

Including antineutrinos

$$H = H_\nu + H_{\bar{\nu}} + H_{\nu\nu} + H_{\bar{\nu}\bar{\nu}} + H_{\nu\bar{\nu}}$$

Requires introduction of a second set of SU(2) algebras!

Including three flavors

Requires introduction of SU(3) algebras.

Both extensions are straightforward, but tedious!
Balantekin and Pehlivan, J. Phys. G **34**, 1783 (2007).

This problem is "exactly solvable" in the single-angle approximation

$$H = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p + \frac{\sqrt{2} G_F}{V} \sum_{\mathbf{p},\mathbf{q}} (1 - \cos \vartheta_{\mathbf{pq}}) \, \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}$$

$$\Downarrow$$

$$H = \sum_p \omega_p \vec{B} \cdot \vec{J}_p + \mu(r) \vec{J} \cdot \vec{J}$$

Note that this Hamiltonian commutes with $\vec{B} \cdot \sum_p J_p$.

Hence $\mathrm{Tr}\left(\rho \vec{B} \cdot \sum_p J_p\right)$ is a constant of motion.

In the mass basis this is equal to $\mathrm{Tr}(\rho J_3)$.

Two of the adiabatic eigenstates of this equation are easy to find in the single-angle approximation:

$$H = \sum_p \omega_p \vec{B} \cdot \vec{J}_p + \mu(r) \vec{J} \cdot \vec{J}$$

$$|j, +j\rangle = |N/2, N/2\rangle = |\nu_1, \ldots, \nu_1\rangle$$

$$|j, -j\rangle = |N/2, -N/2\rangle = |\nu_2, \ldots, \nu_2\rangle$$

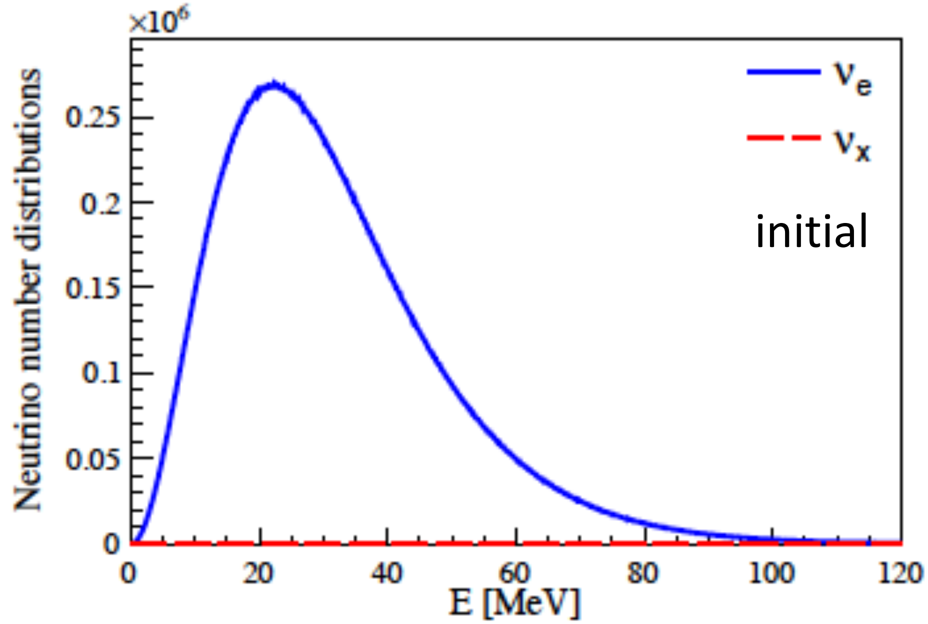
$$E_{\pm N/2} = \mp \sum_p \omega_p \frac{N_p}{2} + \mu \frac{N}{2} \left(\frac{N}{2} + 1 \right)$$

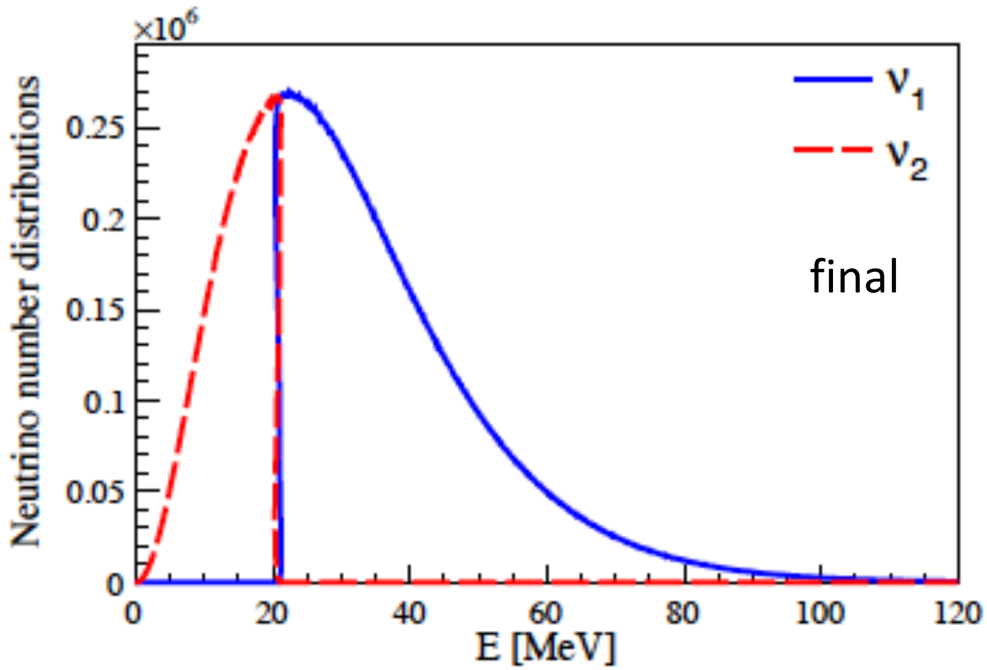
To find the others will take a lot more work

Away from the mean-field:
Adiabatic solution of the *exact* many-body Hamiltonian for extremal states

Adiabatic evolution of an initial thermal distribution (T = 10 MeV) of electron neutrinos. 10^8 neutrinos distributed over 1200 energy bins with solar neutrino parameters and normal hierarchy.

Birol, Pehlivan, Balantekin, Kajino
arXiv:1805.11767
PRD**98** (2018) 083002





BETHE ANSATZ

Single-angle approximation Hamiltonian:

$$H = \sum_p \frac{\delta m^2}{2p} J_p^0 + 2\mu \sum_{\substack{p,\,q \\ p \neq q}} \mathbf{J}_p \bullet \mathbf{J}_q$$

Eigenstates:

$$|x_i\rangle = \prod_{i=1}^N \sum_k \frac{J_k^\dagger}{\left(\delta m^2/2k\right) - x_i} |0\rangle$$

$$\underbrace{-\frac{1}{2\mu} - \sum_k \frac{j_k}{\left(\delta m^2/2k\right) - x_i} = \sum_{j \neq i} \frac{1}{x_i - x_j}}_{\text{Bethe ansatz equations}}$$

$$\mu = \frac{G_F}{\sqrt{2}V}\langle 1 - \cos\Theta \rangle$$

Invariants:

$$h_p = J_p^0 + 2\mu \sum_{\substack{p,\,q \\ p \neq q}} \frac{\mathbf{J}_p \bullet \mathbf{J}_q}{\delta m^2 \left(\frac{1}{p} - \frac{1}{q}\right)}$$

Pehlivan, ABB, Kajino, & Yoshida
Phys. Rev. D 84, 065008 (2011)

A system of N particles each of which can occupy k states (k = number of flavors)

Exact Solution → Mean-field approximation

Entangled and unentangled states → Only unentangled states

Dimension of Hilbert space: k^N

Dimension of the diagonalizing space: kN

von Neumann entropy $S = -\,\mathrm{Tr}\,(\rho \log \rho)$

	Pure State	Mixed State
Density matrix	$\rho^2 = \rho$	$\rho^2 \neq \rho$
Entropy	$S = 0$	$S \neq 0$

Pick one of the neutrinos and introduce the reduced density matrix for this neutrino (with label "b")

$$\tilde{\rho} = \rho_b = \sum_{a,c,d,\ldots} \langle \nu_a, \nu_c, \nu_d, \cdots | \rho | \nu_a, \nu_c, \nu_d, \cdots \rangle$$

Entanglement entropy

$$S = -\mathrm{Tr}\left(\tilde{\rho}\ \log \tilde{\rho}\right)$$

$$\tilde{\rho} = \frac{1}{2}(\mathbb{I} + \vec{\sigma} \cdot \vec{P})$$

$$S = -\frac{1-|\vec{P}|}{2} \log\left(\frac{1-|\vec{P}|}{2}\right) - \frac{1+|\vec{P}|}{2} \log\left(\frac{1+|\vec{P}|}{2}\right)$$

Techniques to solve the exact evolution

- Bethe ansatz method has numerical instabilities for larger values of N. However, it is very valuable since it leads to the identification of conserved quantities.

  Patwardhan *et. al.*, PRD **99**, 123013 (2019); Cervia *et al.*, PRD **100**, 083001 (2019)

- Runge Kutta method (RK4)

  Patwardhan *et. al.*, PRD **104**, 123035 (2021), Siwach *et. al.* PRD **107**, 023019 (2023)

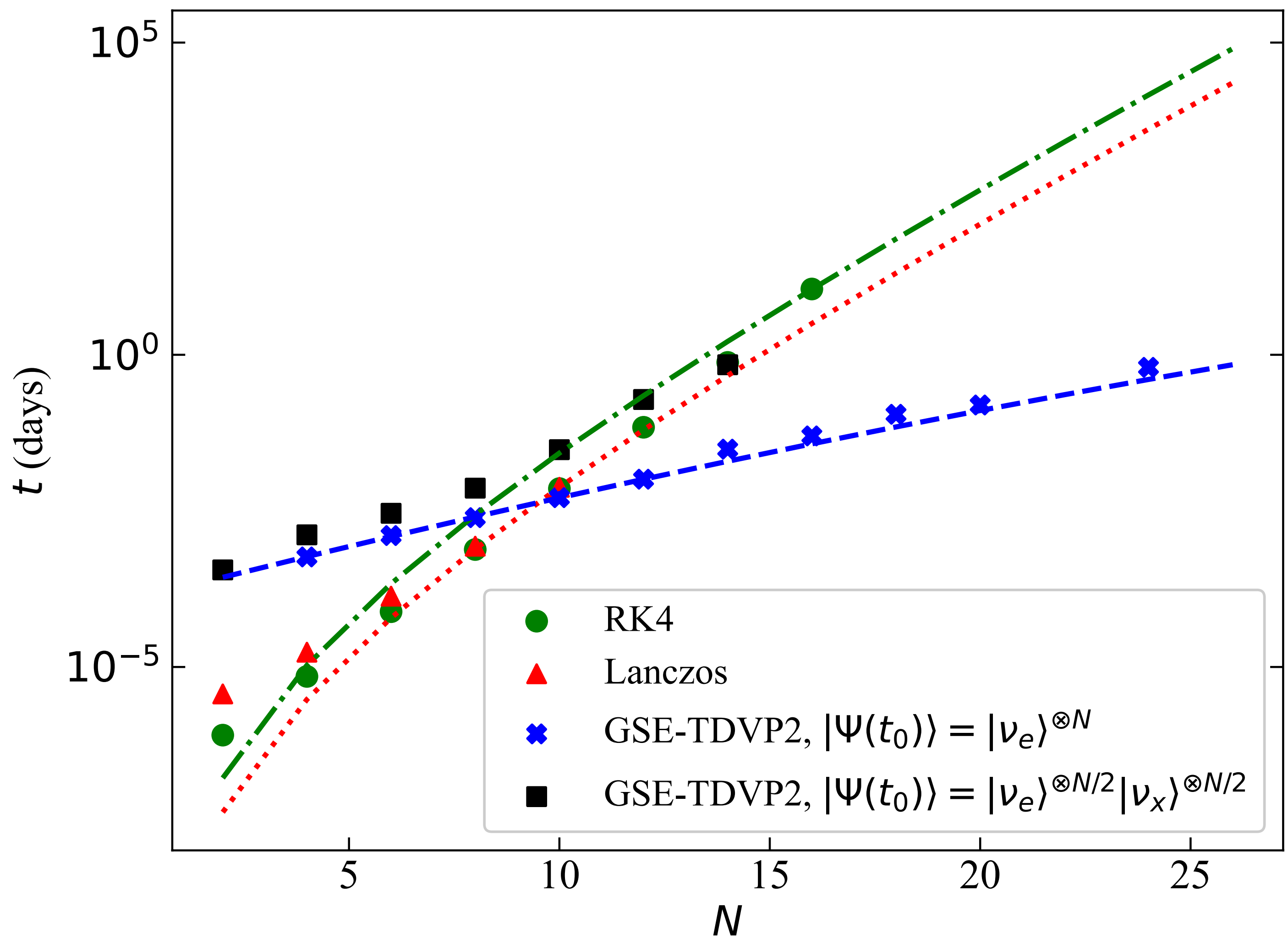
- Tensor network techniques

  Cervia *et al.*, PRD **105**, 123025 (2022)

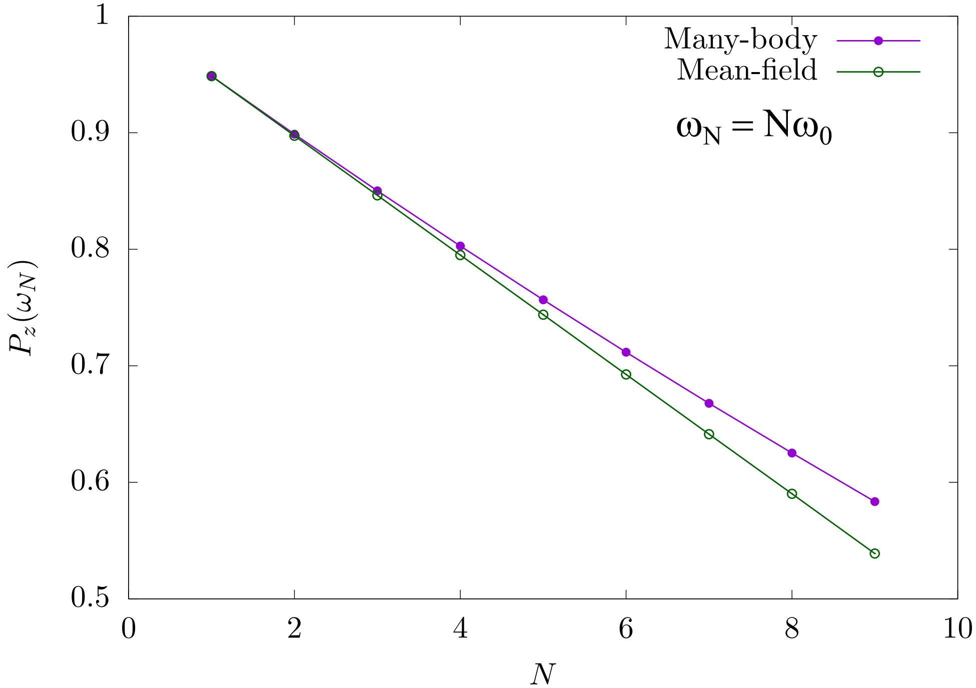
- Noisy quantum computers

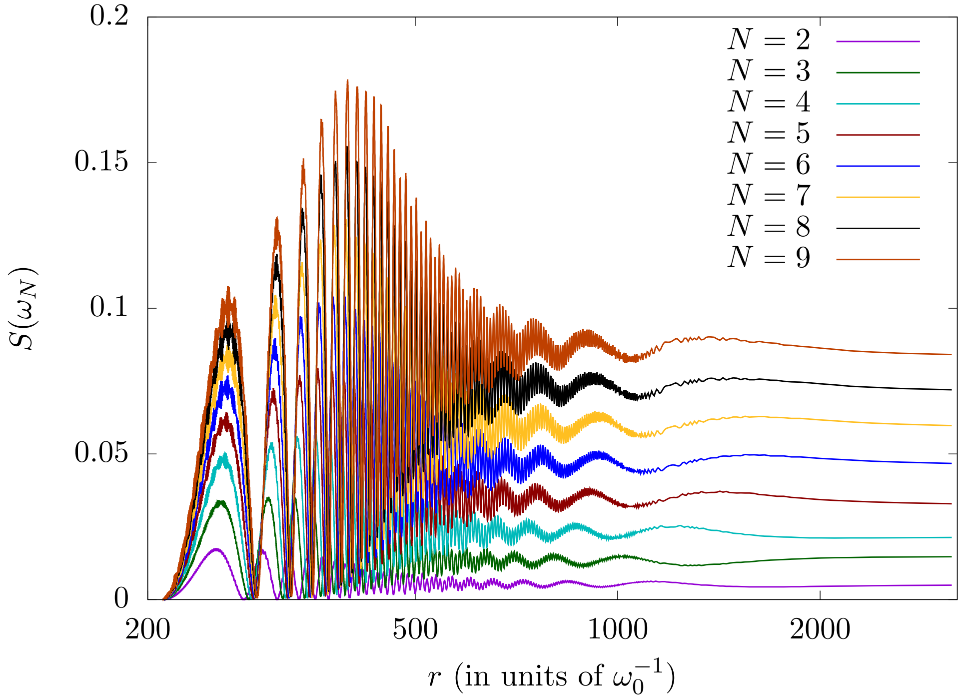
  Siwach *et. al.*, 2308.09123 [quant-ph]

Computation times:



Cervia, Siwach, Patwardhan, Balantekin, Coppersmith, Johnson, arXiv:2202.01865

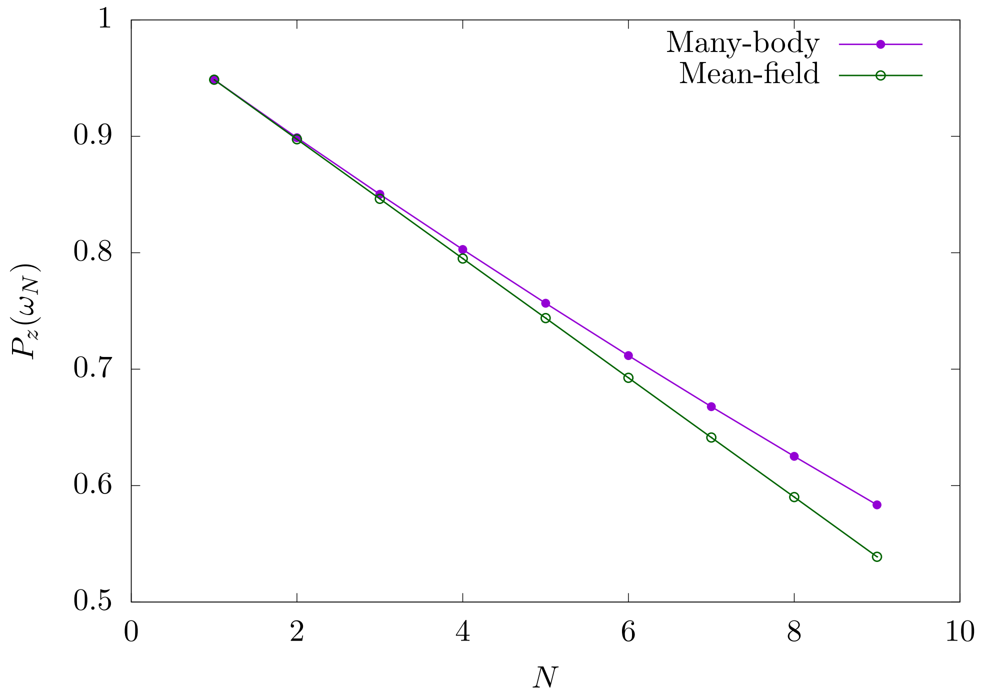




Initial state:
all electron neutrinos

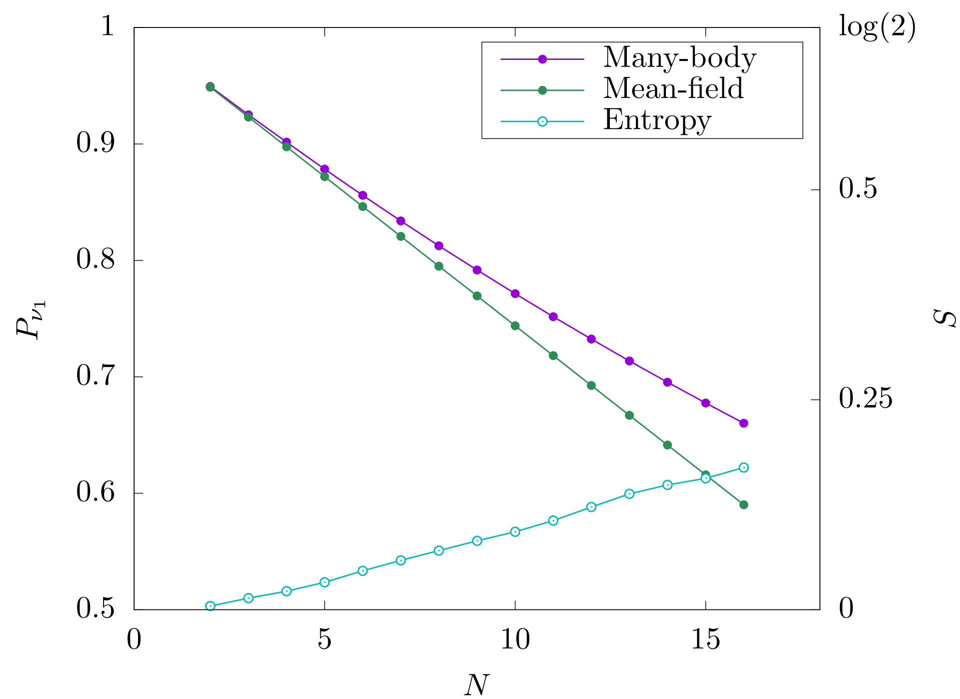
Note: S = 0 for mean-field approximation

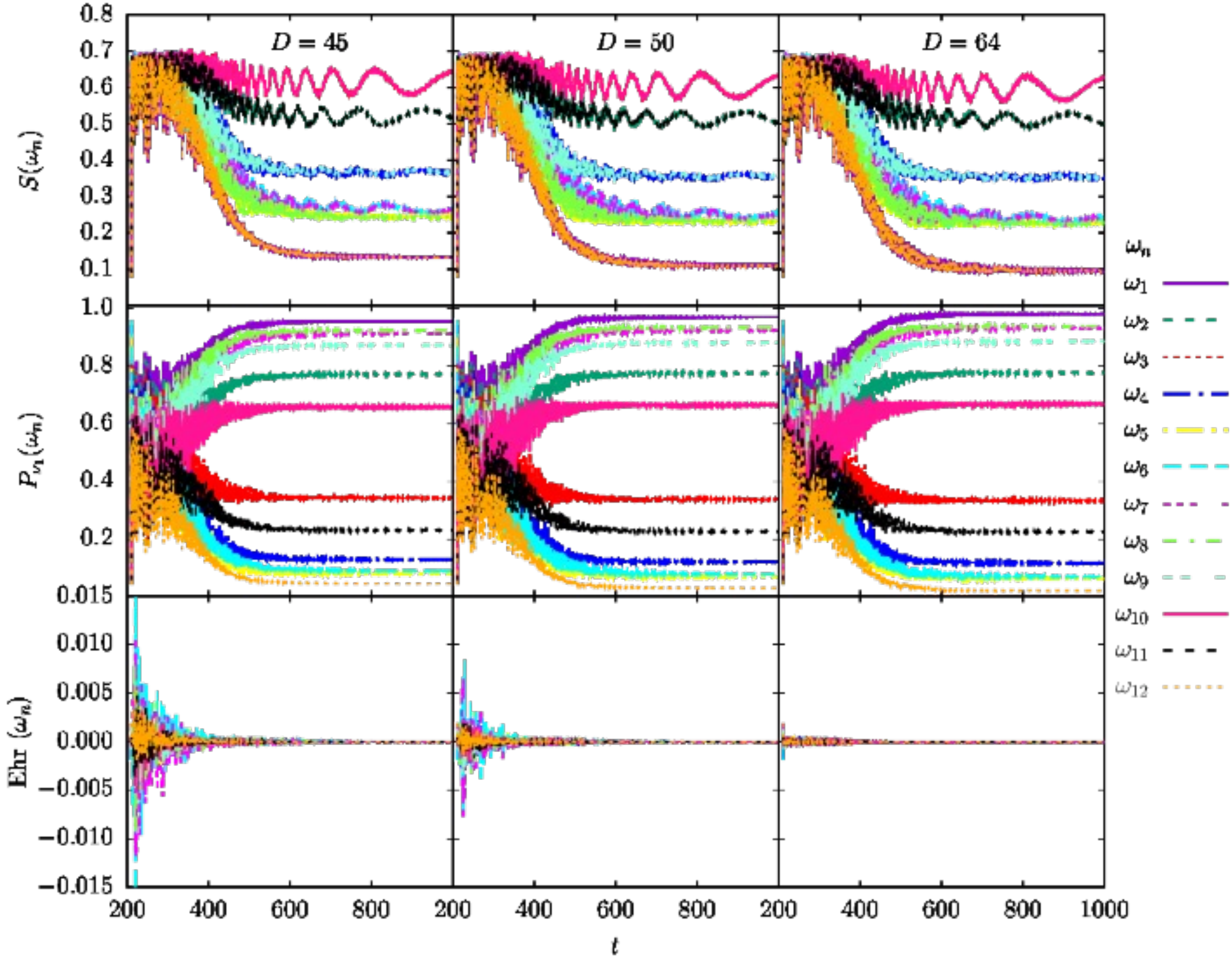
Cervia, Patwardhan, Balantekin, Coppersmith, Johnson, arXiv:1908.03511
PRD, **100**, 083001 (2019)



Cervia, Patwardhan, Balantekin, Coppersmith, Johnson,
arXiv:1908.03511
PRD **100**, 083001 (2019)

Patwardhan, Cervia, Balantekin,
arXiv:2109.08995
PRD **104**, 123035 (2021)





Time evolution for 12 neutrinos (initially six ν_e and six ν_x). D is the bond dimension. The largest possible value of D is 2^6=64.

Mean Field: $\rho = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_N$

$$\omega_A = \frac{\delta m^2}{2E_A}$$

P = Tr (ρ**J**)

$$\rho_A = \frac{1}{2}\left(1 + \vec{\sigma}\cdot\vec{P}^{(A)}\right)$$

$$\frac{\partial}{\partial t}\mathsf{P}^{(A)} = (\omega_A \mathcal{B} + \mu \mathsf{P}) \times \mathsf{P}^{(A)}$$

$$\mathsf{P} = \sum_A \mathsf{P}^{(A)}.$$

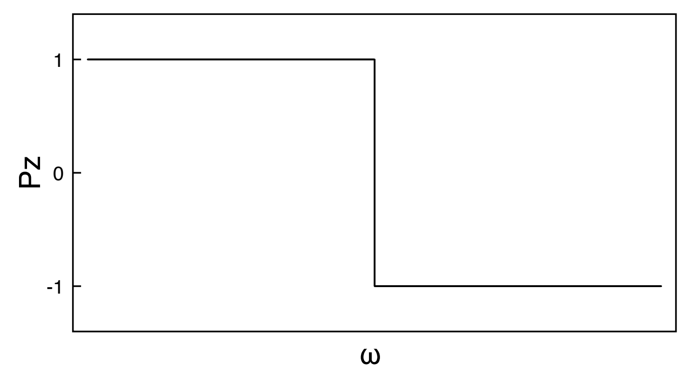
Adiabatic Solution: Each $\mathsf{P}^{(A)}$ lie mostly on the plane defined by $\mathcal{B}$ and P with a small component perpendicular to that plane.

$$\mathrm{P}^{(\mathrm{A})} = \alpha_{\mathrm{A}}\mathcal{B} + \beta_{\mathrm{A}}\mathrm{P} + \gamma_{\mathrm{A}}(\mathcal{B}\times\mathrm{P})$$

Adopt for the mass basis and define $\Gamma = \left(\sum_A \gamma_A \omega_A\right)$. Unless Γ is positive the solutions for P_x and P_y exponentially grow.

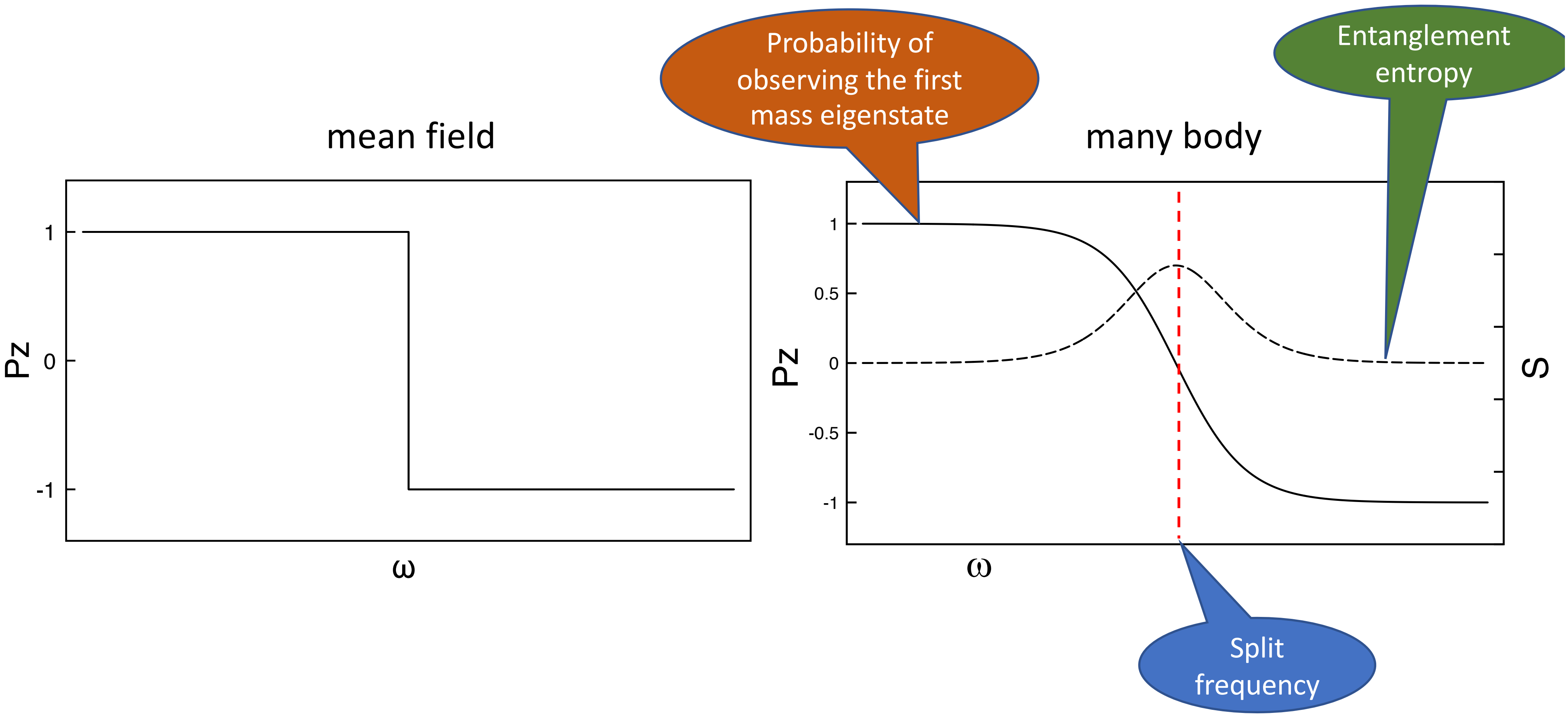
$$P_{x,y} = \Pi_{x,y} \exp\left(-\int \Gamma(t)dt\right)$$

$$\frac{\partial}{\partial t}\Pi_x = \left(\sum_A \beta_A \omega_A\right)\Pi_y, \quad \frac{\partial}{\partial t}\Pi_y = -\left(\sum_A \beta_A \omega_A\right)\Pi_x.$$

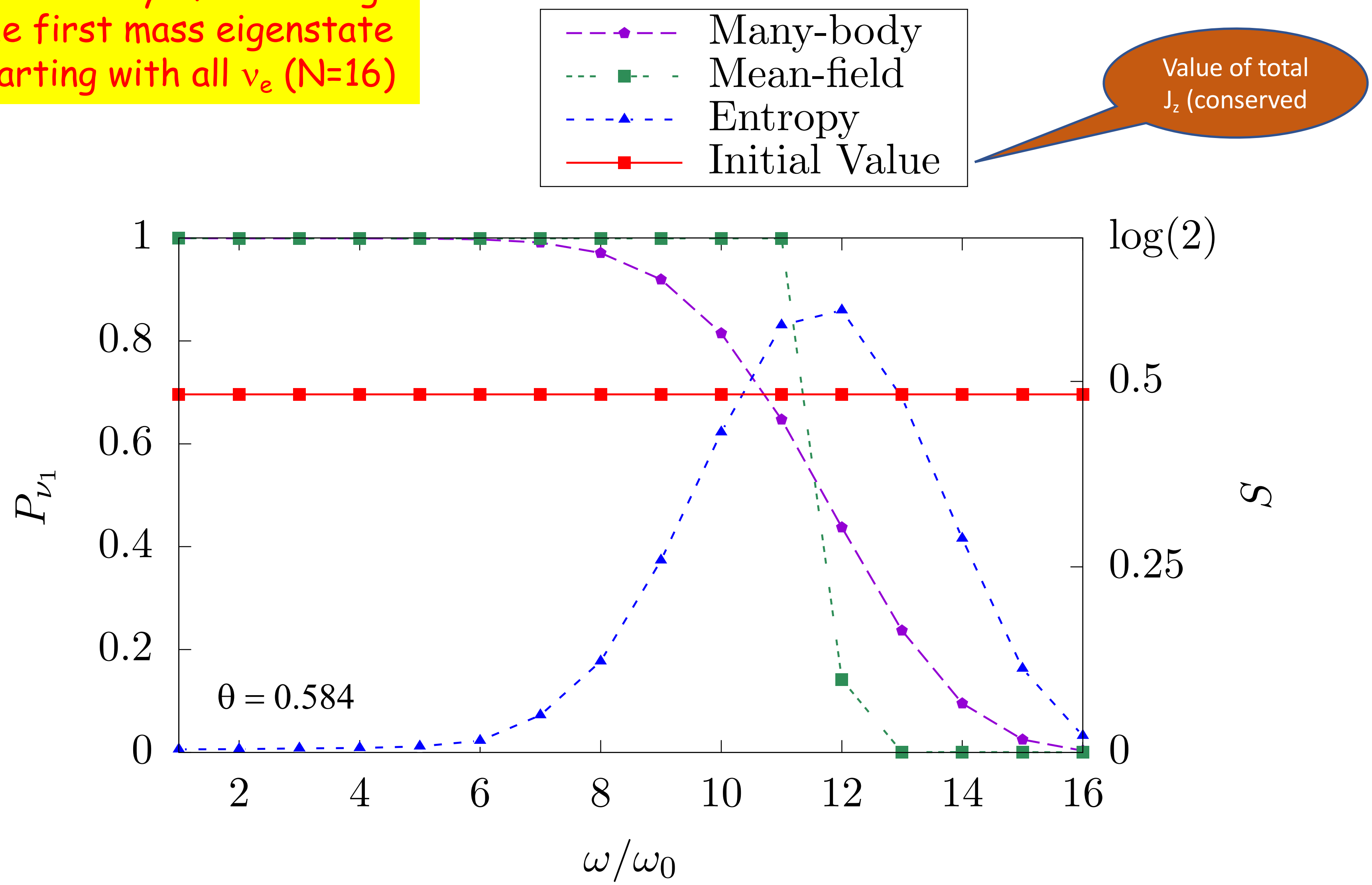


Hence asymptotically P_x and P_y go to zero. Since P^2 is one (uncorrelated neutrinos) $(P_z)^2$ goes to one.

We find that the presence of spectral splits is a good proxy for deviations from the mean-field results

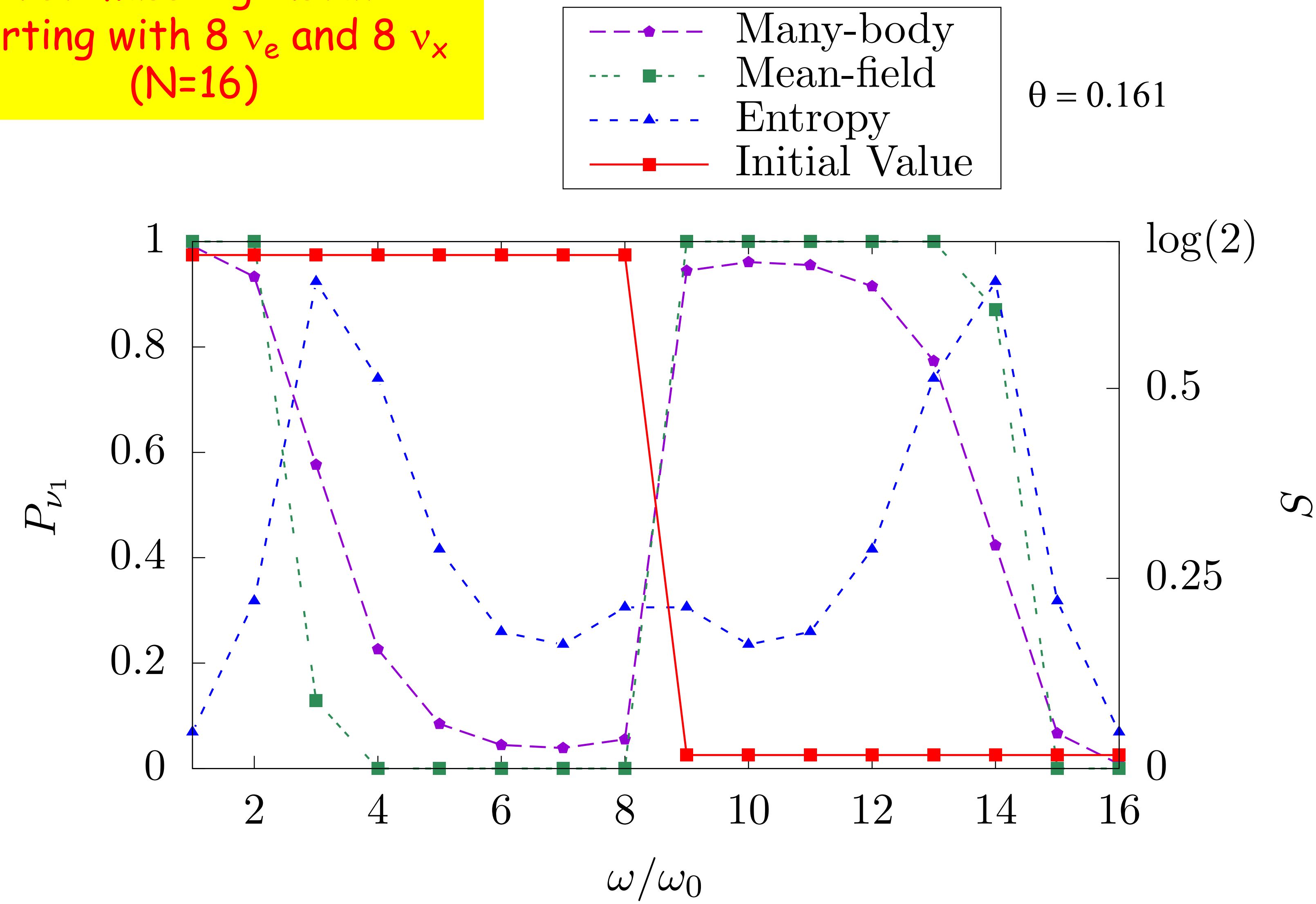


Probability of observing the first mass eigenstate starting with all ν_e (N=16)



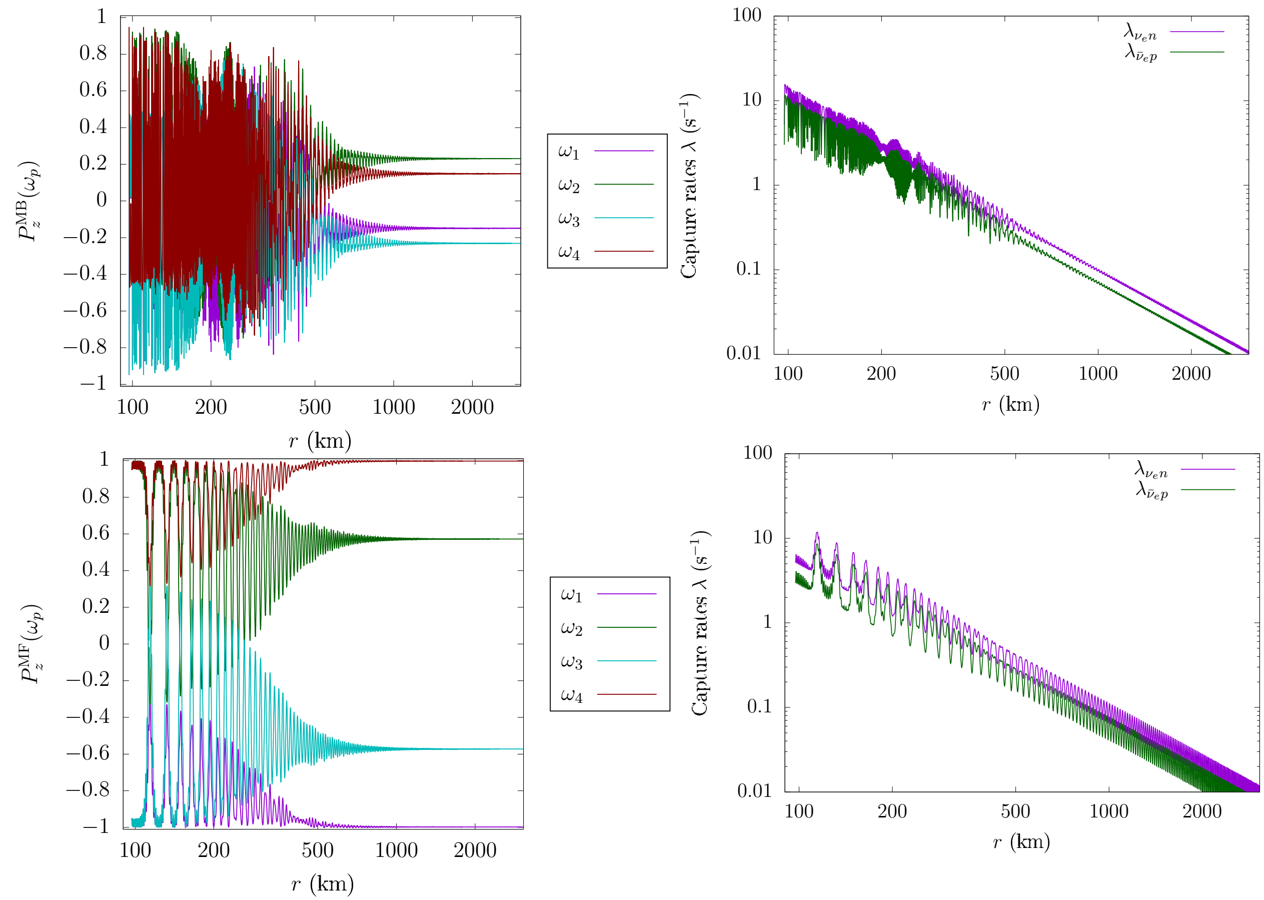
Patwardhan, Cervia, Balantekin, arXiv:2109.08995
Phys. Rev. D **104**, 123035 (2021)

Probability of observing the first mass eigenstate starting with 8 ν_e and 8 ν_x (N=16)



Patwardhan, Cervia, Balantekin, arXiv:2109.08995

The impact of two different treatments of collective neutrino oscillations (with and without entanglement)



$$\omega_i = \frac{\delta m^2}{2E_i}$$

ω_1: $\bar{\nu}_e$, ω_2: $\bar{\nu}_x$, ω_3 : ν_x, ω_4: ν_e

Balantekin, Cervia, Patwardhan, Surman, Wang; 2311.02562 [astro-ph.HE]

Considerations of collective effects unveiled a new kind of nucleosynthesis: “The νi process”.

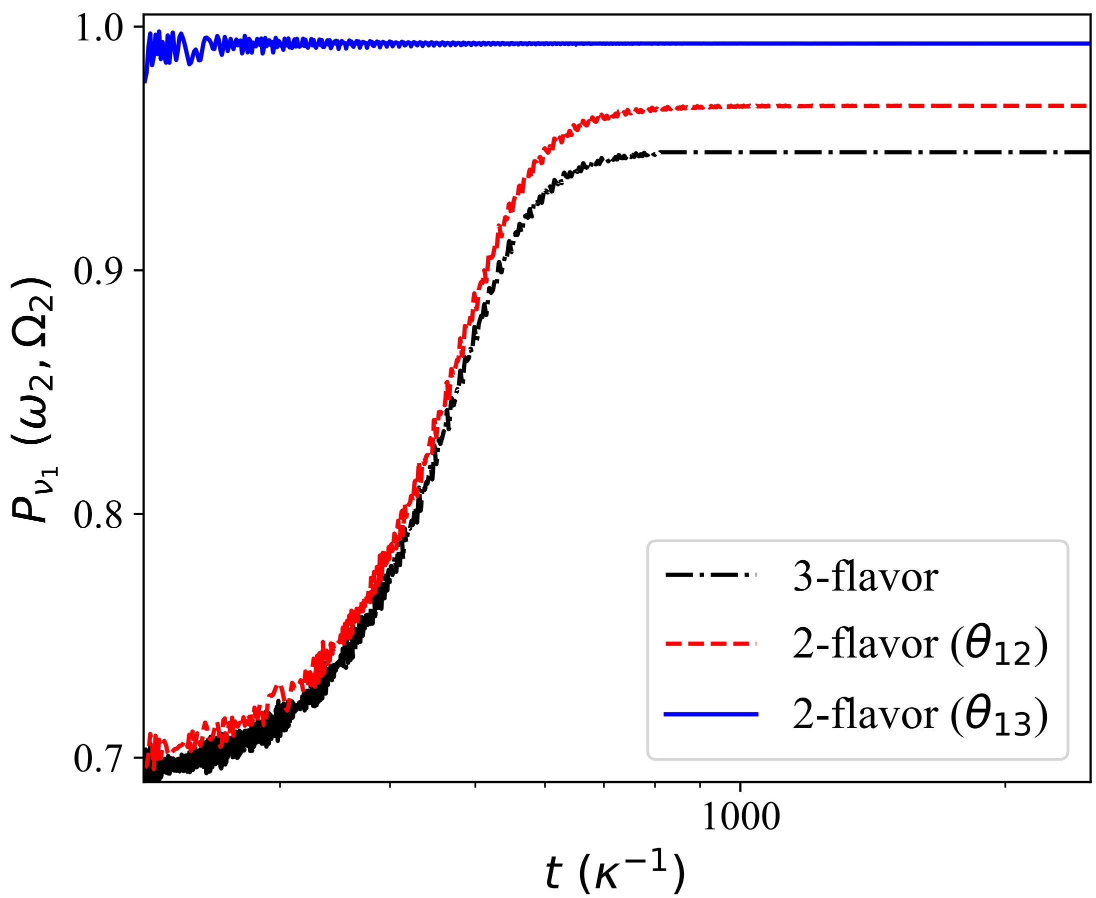
See Rebecca Surman’s talk!

Entanglement in three-flavor collective oscillations

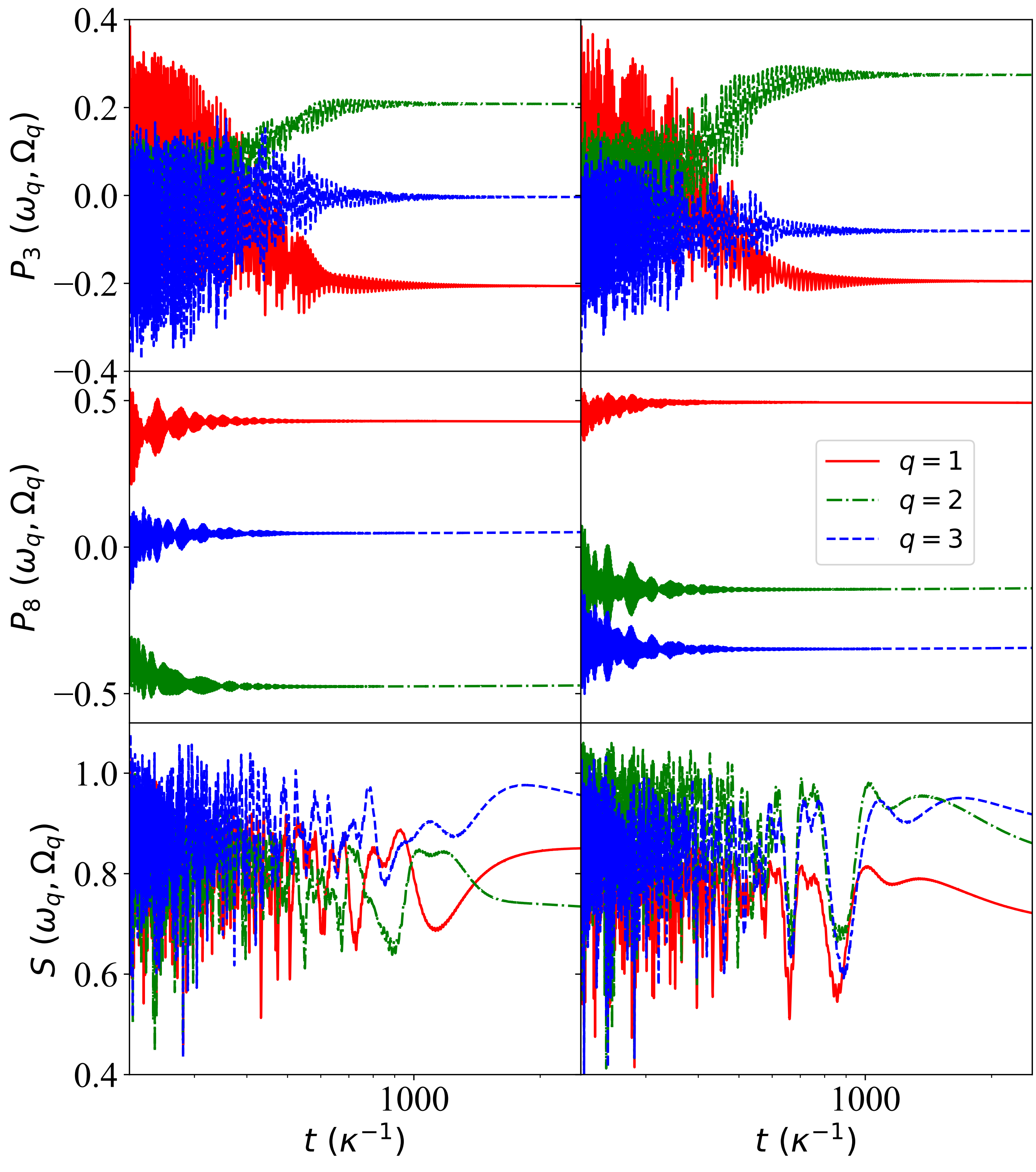
$$H = \sum_p \vec{B} \cdot \vec{Q}(p) + \sum_{p,k} \mu_{pk}\, \vec{Q}(p) \cdot \vec{Q}(k)$$

$$Q_A(p) = \frac{1}{2} \sum_{i,j=1}^{3} a_i^\dagger(p) (\lambda_A)_{ij} a_j(p)$$

$$B = \frac{1}{2E}\left(0,0, m_1^2 - m_2^2, 0,0,0,0, -\left|m_3^2 - m_1^2\right|\right)$$



Pooja Siwach, Anna Suliga, A.B. Balantekin
Physical Review D **107** (2023) 2, 023019


0.4
0.2
0.0
−0.2
−0.4
$P_3\ (\omega_q, \Omega_q)$
0.5
0.0
−0.5
$P_8\ (\omega_q, \Omega_q)$
$q = 1$
$q = 2$
$q = 3$
1.0
0.8
0.6
0.4
$S\ (\omega_q, \Omega_q)$
1000
1000
$t\ (\kappa^{-1})$
$t\ (\kappa^{-1})$

Qutrits are more complicated than qubits

Density matrix for a single qutrit $\rho = \frac{1}{3}(1 + \lambda_i P_i)$

$P_i P_i \leq 3$
$Q = d_{ijk} P_i P_j P_k$

Positive semi-definite condition

Pure state only if $P^2 = P_i P_i = 3$ and $P_i = d_{ijk} P_j P_k$

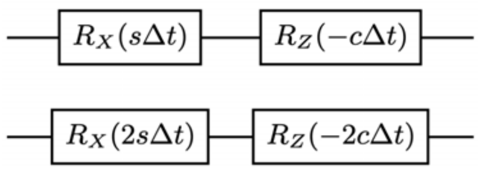
In the above equations d_{ijk} is the completely symmetric tensor of SU(3). Note the duality between SU(3) Casimir operators and invariants of the density matrix.

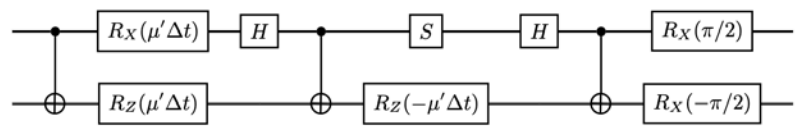
A.B. Balantekin and A. Suliga, in preparation

Putting on a quantum computer

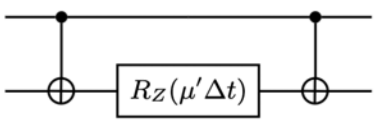
First try: Brute Force – simple trotterization for two neutrinos and two flavors

$$H = \frac{1}{2}[\sin\theta\, X_0 - \cos\theta\, Z_0 + 2\sin\theta\, X_1 - 2\cos\theta\, Z_1 + \mu(t)(X_0X_1 + Y_0Y_1 + Z_0Z_1)]$$



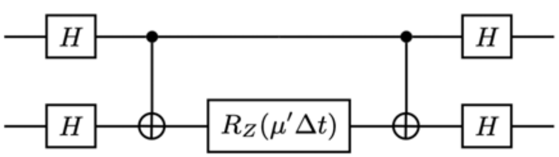


$$e^{-i\frac{\mu\prime}{2}\Delta t(X_0X_1+Y_0Y_1+Z_0Z_1)}$$



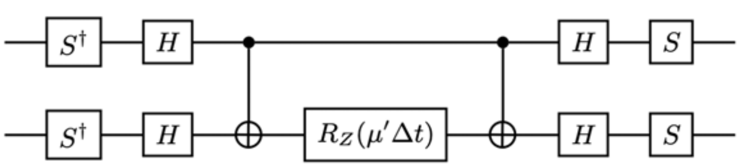
(a)

$$e^{-i\frac{\mu\prime}{2}\Delta t(Z_0Z_1)}$$



(b)

$$e^{-i\frac{\mu\prime}{2}\Delta t(X_0X_1)}$$



(c)

$$e^{-i\frac{\mu\prime}{2}\Delta t(Y_0Y_1)}$$

Reduced number of CNOT gates

Even for only two neutrinos and after reducing the number of CNOT gates, the circuits remain too deep. We then adopt a hybrid approach of Bharti and Haug, PRA **104**, 042418 (2022).

The hybrid approach of Bharti and Haug, PRA **104**, 042418 (2022).

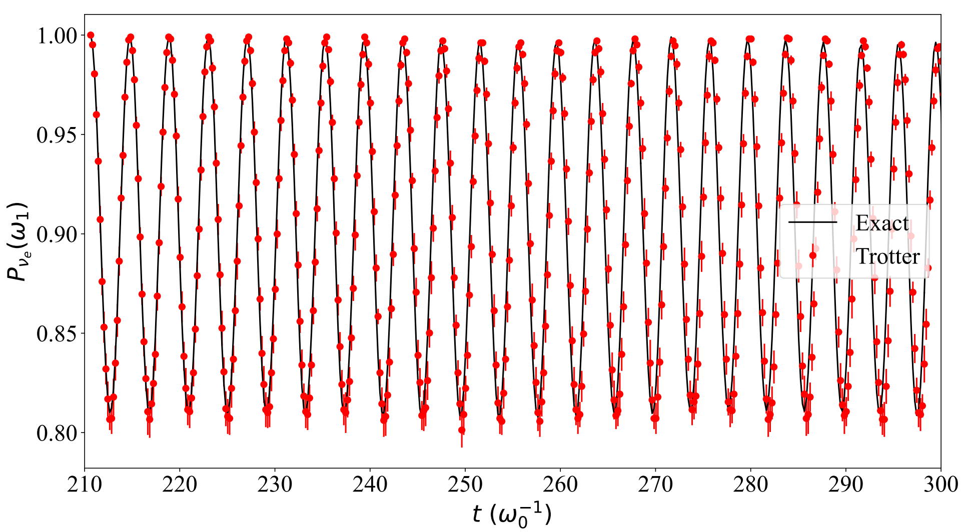
Hamiltonian is a sum of unitaries $H = \sum_{i=1}^{r} \beta_i U_i$

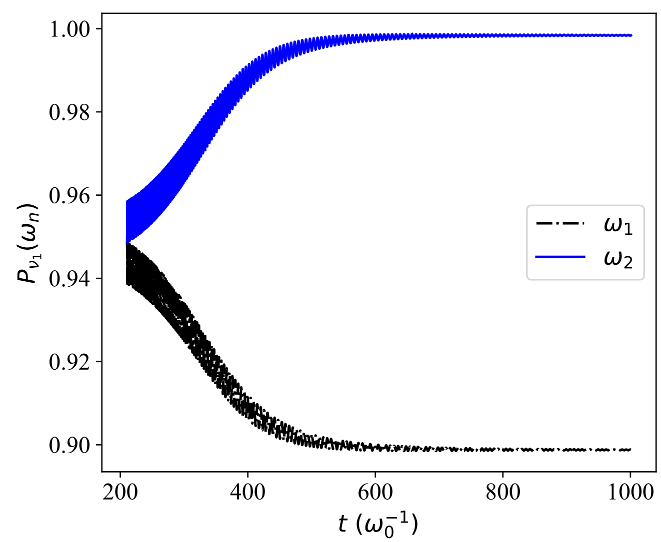
Ansatz for the state $|\phi(\alpha(t))\rangle = \sum_{i=1}^{r} \alpha_i(t)|\psi_i\rangle$

$\langle\psi_i|\psi_j\rangle = \varepsilon_{ij}$ $\quad \alpha^\dagger \varepsilon\, \alpha = 1$ $\quad D_{ij} = \sum_k \beta_k \langle\psi_i|U_k|\psi_j\rangle$ $\quad i\varepsilon\frac{\partial\alpha}{\partial t} = D\alpha(t)$

Choose three basis states $|\psi_1\rangle = X_0|00\rangle,\quad |\psi_2\rangle = X_1|00\rangle,\quad |\psi_3\rangle = X_0X_1|00\rangle$

ε and D are calculated using a quantum computer, rest is done on a classical computer





P. Siwach, K. Harrison, and A.B. Balantekin, Phys. Rev. D **108** (2023) 8, 083039

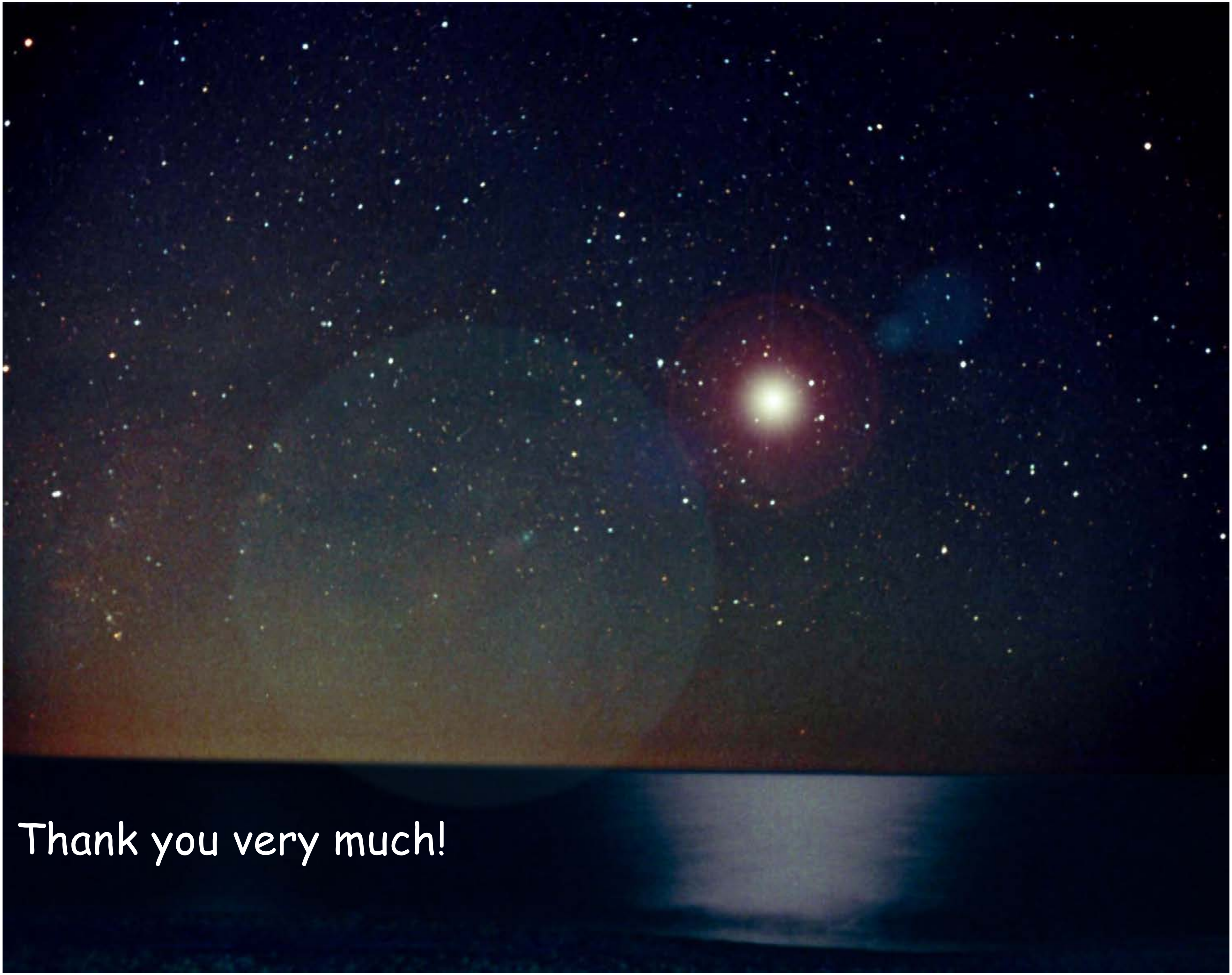
A group of us (Cervia, Patwardhan, Roggero, Rrapaj, Siwach, myself and few others), as a first step, are performing circuit analyses for the time evolution as a function of particle number for both qubit and qutrit implementations of the three flavor systems.

See Ermal Rrapaj's talk!

In this and other QC/QIS projects we would most welcome collaborators from RIKEN!

CONCLUSIONS

- Calculations performed using the mean-field approximation have revealed a lot of interesting physics about collective behavior of neutrinos in astrophysical environments. Here we have explored possible scenarios where further interesting features can arise by going beyond this approximation.
- We found that the deviation of the adiabatic many-body results from the mean field results is largest for neutrinos with energies around the spectral split energies. In our single-angle calculations we observe a broadening of the spectral split region. This broadening does not appear in single-angle mean-field calculations and seems to be larger than that was observed in multi-angle mean-field calculations (or with BSM physics).
- (From the QIS perspective) For simplicity originally two neutrino flavors were mapped onto qubits. But since neutrinos come in three flavors, neutrinos should be mapped onto qutrits. The description of qutrits is much more involved than that of qubits.


Thank you very much!

EXTRA SLIDES

This is a growing field, a partial list of other work:

- E. Rrapaj, Phys. Rev. C 101, 065805 (2020).
- Roggero, Phys. Rev. D 104(10), 103016 (2021).
- A. Roggero, E. Rrapaj, Z. Xiong, Phys. Rev. D 106, 043022 (2022).
- Z. Xiong, Phys. Rev. D 105(10), 103002 (2022).
- J.D. Martin, A. Roggero, H. Duan, J. Carlson, V. Cirigliano, Phys. Rev. D 105(8), 083020 (2022).
- D. LaCroix et.al., Phys.Rev.D 106 (2022) 12, 123006
- M. Illa and M.J. Savage, Phys.Rev.Lett. 130 (2023) 22, 221003

Many neutrino system

This is the only many-body system driven by the weak interactions:

Table: Many-body systems

Nuclei	Strong	at most $\sim$250 particles
Condensed matter	E&M	at most N_A particles
ν**'s in SN**	Weak	$\sim 10^{58}$particles

Astrophysical extremes allow us to test physics that cannot be tested elsewhere!

Note that if you have N neutrinos, you do not only have total $j=N/2$, but you have total $j = N/2, (N/2)-1, (N/2)-2$, etc. You can not deduce the properties of an N neutrino system by studying $j = N/2$!

Example:

N neutrinos: true size of the Hilbert Space = 2^N

$J=N/2$: size of the Hilbert Space = $2j+1 = N+1$

A severe truncation!

$$
H_{\nu\nu} = \frac{G_F}{\sqrt{2}V}\int d^3p\, d^3q\left(1-\cos\theta_{\vec{p}\cdot\vec{q}}\right)\left[a_e^\dagger(p)a_e(p)a_e^\dagger(q)a_e(q) + a_x^\dagger(p)a_x(p)a_x^\dagger(q)a_x(q) + a_x^\dagger(p)a_e(p)a_e^\dagger(q)a_x(q) + a_e^\dagger(p)a_x(p)a_x^\dagger(q)a_e(q)\right]
$$

$$
J_+(p) = a_x^\dagger(p)a_e(p),\ J_-(p) = a_e^\dagger(p)a_x(p),\ J_0(p) = \frac{1}{2}\left(a_x^\dagger(p)a_x(p) - a_e^\dagger(q)a_e(q)\right)
$$

$$
H_{\nu\nu} = \left(\quad\right)\left[N^2 - \left(\int d^3p\frac{\vec{p}}{|\vec{p}|}N(p)\right)\cdot\left(\int d^3p\frac{\vec{p}}{|\vec{p}|}N(p)\right)\right] + \frac{\sqrt{2}G_F}{V}\int d^3p\, d^3q\left(1-\cos\theta_{\vec{p}\cdot\vec{q}}\right)\vec{J}(p)\cdot\vec{J}(\mathrm{q})
$$

$$H_{\nu\nu} = (\quad)\left[N^2 - \left(\int d^3p \frac{\vec{p}}{|\vec{p}|} N(p)\right) \cdot \left(\int d^3p \frac{\vec{p}}{|\vec{p}|} N(p)\right)\right] + \frac{\sqrt{2}G_F}{V} \int d^3p\, d^3q \left(1 - \cos\theta_{\vec{p}\cdot\vec{q}}\right) \vec{J}(p) \cdot \vec{J}(\mathrm{q})$$

Concerns were raised recently about the terms proportional to N (p). However, these terms do not contribute to the quantum evolution since

$$[N, H_\nu] = 0 = [N, \vec{J}(p) \cdot \vec{J}(\mathrm{q})]$$

$$\widehat{U} = e^{-i(\quad)tN - iN^2 \int dt\, \mu}\, \widehat{V}$$

V includes terms independent of N. Hence

$$\rho = \widehat{U} \rho_i \widehat{U}^\dagger = \widehat{V} \rho_i \widehat{V}^\dagger$$

$$H_{\nu\nu} = \frac{\sqrt{2}G_F}{V}\int d^3p\, d^3q\left(1 - \cos\theta_{\vec{p}\cdot\vec{q}}\right)\vec{J}(p)\cdot\vec{J}(\mathrm{q})$$

How do we get the mean-field from this many-body Hamiltonian? Procedure was already given by Balantekin and Pehlivan, J. Phys. G 34, 47 (2007). Introduce SU(2) coherent states (for two-flavors):

$$|z(t)\rangle = \exp\left(-\frac{1}{2}\int d^3p\ \log(1 + |z(p,t)|^2)\right)\exp\left(\int d^3p\, z(p,t)J_+(p)\right)\prod a_e^\dagger|0\rangle$$

Then write the evolution operator in the basis of SU(2) coherent states

$$\left\langle z(t_f)\middle|\hat{U}\middle|z(t_i)\right\rangle = \int \mathcal{D}[z,z^*]e^{-i\mathcal{S}[z,z^*]}$$

$$\mathcal{S}[z,z^*] = \int_{t_i}^{t_f} dt\ \left\langle i\frac{\partial}{\partial t} - H_\nu - H_{\nu\nu}\right\rangle - i\log\left\langle z(t_f)\middle|z(t_f)\right\rangle$$

$$\mathcal{S}[z,z^*] = \int_{t_i}^{t_f} dt \; \underbrace{\left\langle i\frac{\partial}{\partial t} - H_\nu - H_{\nu\nu} \right\rangle}_{\mathcal{L}} - i\log\langle z(t_f)|z(t_f)\rangle$$

We then follow the standard procedure to find the stationary points of this action to obtain the Euler-Lagrange equations:

$$\boxed{\left(\frac{d}{dt}\frac{\partial}{\partial \dot{z}} - \frac{\partial}{\partial z}\right)\mathcal{L}(z,z^*) = 0, \qquad \left(\frac{d}{dt}\frac{\partial}{\partial \dot{z}^*} - \frac{\partial}{\partial z^*}\right)\mathcal{L}(z,z^*) = 0}$$

Solving Euler-Lagrange eqs. gives us the mean-field eqs. with $z = \dfrac{\psi_x}{\psi_e}$

subject to $|\psi_e|^2 + |\psi_x|^2 = 1$

Balantekin and Pehlivan, J. Phys. G **34**, 47 (2007)

How do you find many-body corrections to the mean-field? Expand the action around the stationary phase (mean-field) solution:

$$\mathcal{S}[z,z^*] = \mathcal{S}\left[z_{sp}, z_{sp}^*\right] + \frac{1}{2}\left(z - z_{sp}\right)^T \left(\frac{\delta^2 \mathcal{S}}{\delta z \delta z}\right)_{sp} \left(z - z_{sp}\right) + \left(z - z_{sp}\right)^T \left(\frac{\delta^2 \mathcal{S}}{\delta z \delta z^*}\right)_{sp} \left(z^* - z_{sp}^*\right) + \frac{1}{2}\left(z^* - z_{sp}^*\right)^T \left(\frac{\delta^2 \mathcal{S}}{\delta z^* \delta z^*}\right)_{sp} \left(z^* - z_{sp}^*\right) + \mathcal{O}(z^3)$$

The Gaussian integral is then straightforward to calculate:

$$\left\langle z(t_f) \middle| \hat{U} \middle| z(t_i) \right\rangle = \int \mathcal{D}[z,z^*] e^{-i\mathcal{S}[z,z^*]} \propto \frac{e^{-i\,\mathcal{S}[z_{sp}, z_{sp}^*]}}{\sqrt{\det\left(KM - L^T K^{-1} L\right)}}$$

$$K = \frac{1}{2}\left(\frac{\delta^2 \mathcal{S}}{\delta x \delta x}\right)_{sp} \qquad M = \frac{1}{2}\left(\frac{\delta^2 \mathcal{S}}{\delta y \delta y}\right)_{sp} \qquad L = \frac{1}{2}\left(\frac{\delta^2 \mathcal{S}}{\delta x \delta y}\right)_{sp} \qquad z = x + iy$$

The "pre-exponential" determinant has not been calculated in the most general case. Its calculation in the general case would be the only rigorous way to assess how much many-body case deviates from the mean-field results.

Balantekin and Pehlivan, J. Phys. G **34**, 47 (2007)

Two of the adiabatic eigenstates of this equation are easy to find in the single-angle approximation:

$$H = \sum_p \omega_p \vec{B} \cdot \vec{J}_p + \mu(r) \vec{J} \cdot \vec{J}$$

$$|j, +j\rangle = |N/2, N/2\rangle = |\nu_1, \ldots, \nu_1\rangle$$

$$|j, -j\rangle = |N/2, -N/2\rangle = |\nu_2, \ldots, \nu_2\rangle$$

$$E_{\pm N/2} = \mp \sum_p \omega_p \frac{N_p}{2} + \mu \frac{N}{2} \left(\frac{N}{2} + 1 \right)$$

To find the others will take a lot more work