Quantum Kinetic Equations of Neutrino Oscillations

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Credit: Ewert

Motivation

- Ejecta of binary neutron star merger
  - Collective oscillations can lead to significant flavor conversion of neutrinos in the ejecta, potentially altering the overall composition and affecting nucleosynthesis processes.
  - Help constraint neutron star EOS.
  - Neutrinos from neutron star mergers can potentially offer a complimentary perspective to gravitational wave observations
- Core-collapse supernova
  - Enhance or hinder the explosion?
  - Determining Neutrino Properties, such as their mass hierarchy, CP-violation, possible sterile neutrino and interactions beyond the standard model (dark matter).

"Classical":
Treated as radiation energy transportation
->moment-based methods

"Quantum":
Oscillation+QED effects: absorption and emission, pair production and annihilation, nucleon-nucleon bremsstrahlung radiation, and neutrino-neutrino pair annihilation and scattering.
->quantum kinetics

Theory:	Simulation:
1302.2374	2206.04098
1903.00022	2212.01409
2306.14982	2305.11207

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Neutrino Oscillation

- Theory

Neutrinos have different flavors and masses, the flavor and mass state of neutrinos are connected by

$$\left|\nu_\alpha\right\rangle = \sum_k U^*_{\alpha k}\left|\nu_k\right\rangle \ (\alpha = e, \mu, \tau)$$

where the $U^*_{\alpha k}$ is the mixing matrix (i.e. if not diagonal, the neutrinos are mixed) and $\left|\nu_k\right\rangle$ is the massive state. The massive state are eigenstates of the Hamiltonian

$$H\left|\nu_k\right\rangle = E_k\left|\nu_k\right\rangle$$

with energy eigenvalues

$$E_k = \sqrt{\vec{p}^{\,2} + m_k^2}$$

The Schrödinger equation

$$i\frac{\mathrm{d}}{\mathrm{d}t}\left|\nu_k(t)\right\rangle = H\left|\nu_k(t)\right\rangle$$

implies that the massive neutrino states evolve in time as plane waves

$$\left|\nu_k(t)\right\rangle = e^{-iE_k t}\left|\nu_k\right\rangle$$

Let us consider now a flavor state $\left|\nu_\alpha(t)\right\rangle$ which describes a neutrino created with a definite flavor α at time $t = 0$. The time evolution of this state is given by

$$\left|\nu_\alpha(t)\right\rangle = \sum_k U^*_{\alpha k} e^{-iE_k t}\left|\nu_k\right\rangle$$

Neutrino Oscillation

- Theory

Using the unitarity relation

$$U^\dagger U = \mathbf{1} \iff \sum_\alpha U^*_{\alpha k} U_{\alpha j} = \delta_{jk}$$

the massive states can be expressed in terms of flavor states $\left|\nu_k\right\rangle = \sum_\alpha U_{\alpha k}\left|\nu_\alpha\right\rangle$. So, we obtain

$$\left|\nu_\alpha(t)\right\rangle = \sum_{\beta=e,\mu,\tau}\left(\sum_k U^*_{\alpha k} e^{-iE_k t} U_{\beta k}\right)\left|\nu_\beta\right\rangle$$

Hence the probability of $\nu_\alpha \to \nu_\beta$ transition as a function of time is, then, given by

$$P_{\nu_\alpha \to \nu_\beta}(t) \equiv \left\langle \nu_\beta \mid \nu_\alpha(t)\right\rangle = \left|A_{\nu_\alpha \to \nu_\beta}(t)\right|^2 = \sum_{k,j} U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} e^{-i\left(E_k - E_j\right)}$$

For relativistic neutrinos, $E_k \simeq E + \dfrac{m_k^2}{2E}$, such that $E_k - E_j \simeq \dfrac{\Delta m^2_{kj}}{2E}$, where $E = |\vec{p}\,|$. Therefore, the transition probability can be approximated by

$$P_{\nu_\alpha \to \nu_\beta}(t) = \sum_{k,j} U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} \exp\left(-i\frac{\Delta m^2_{kj} t}{2E}\right).$$

Since neutrinos are ultra-relativistic particles, which travel at about the speed of light, we can approximately write $t = L$, leading to

$$P_{\nu_\alpha \to \nu_\beta}(L, E) = \sum_{k,j} U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} \exp\left(-i\frac{\Delta m^2_{kj} L}{2E}\right)$$

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Neutrino Oscillation

- Theory

This expression shows that the source-detector distance L and the neutrino energy E are the quantities depending on the experiment which determine the phases of neutrino oscillations

$$\Phi_{kj} = -\frac{\Delta m_{kj}^2 L}{2E}$$

Sometimes it is convenient to write the probability as

$$P_{\nu_\alpha \to \nu_\beta}(L,E) = \sum_k \left|U_{\alpha k}\right|^2 \left|U_{\beta k}\right|^2 + 2\mathfrak{Re} \sum_{k>j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left(-2\pi i \frac{L}{L_{kj}^{\mathrm{osc}}}\right),$$

so that we can separate a constant term from the oscillating term, which is characterized by the oscillation lengths

$$L_{kj}^{\mathrm{osc}} = \frac{4\pi E}{\Delta m_{kj}^2}.$$

The oscillation length L_{kj}^{osc} is the distance at which the phase becomes equal to 2π.

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Neutrino Oscillation

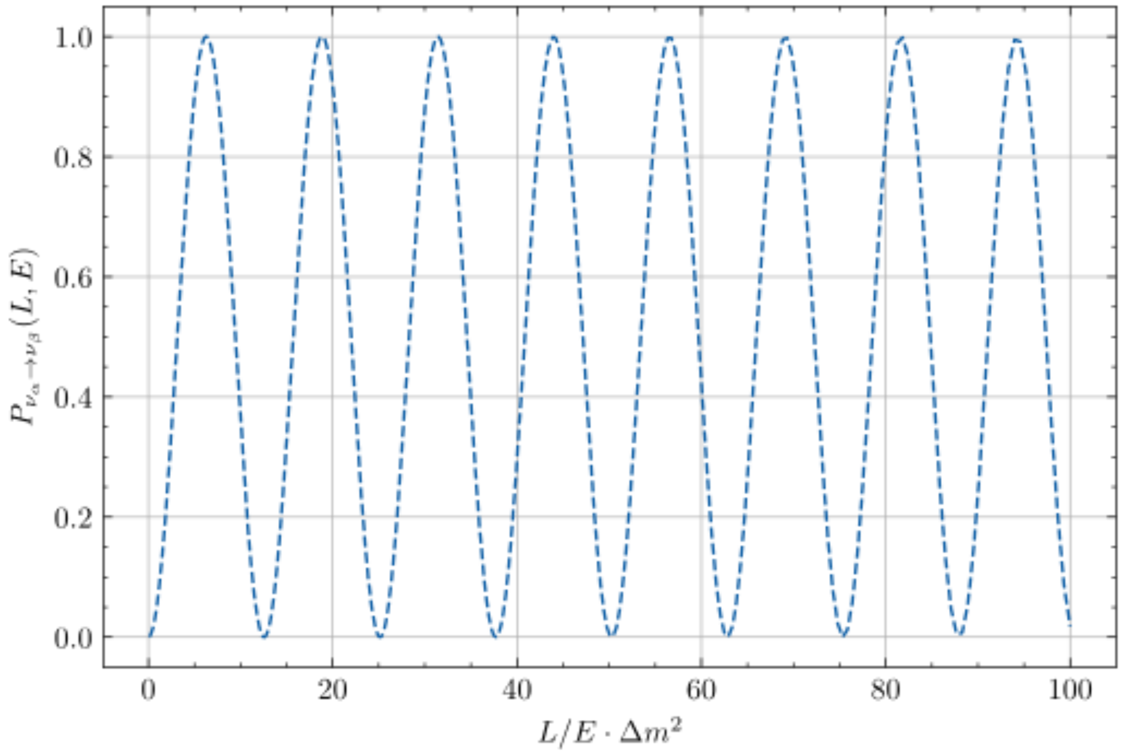
- Example 1

Now we just consider one example of neutrino oscillation which we mix two neutrino flavors α and β. Assume that the mixing matrix is

$$U = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$$

in which ϑ is the mixing angle, with a value in the interval $0 \leq \vartheta \leq \pi/2^2$. Therefore, the probability of $\nu_\alpha \to \nu_\beta$ transitions with $\alpha \neq \beta$ can be easily derived

$$P_{\nu_\alpha \to \nu_\beta}(L, E) = \frac{1}{2}\sin^2 2\vartheta \left[1 - \cos\left(\frac{\Delta m^2 L}{2E}\right)\right] = \sin^2 2\vartheta \sin^2\left(\frac{\Delta m^2 L}{4E}\right) \quad (\alpha \neq \beta)\,.$$



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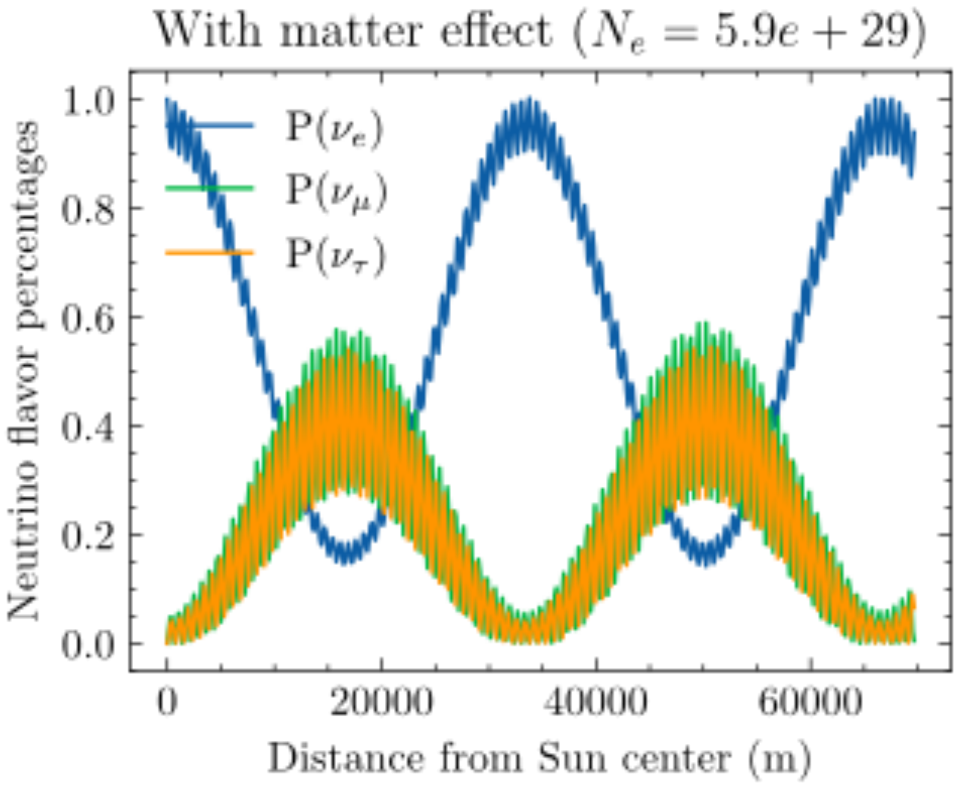
Neutrino Oscillation

- Example 2

The matter potential in the local comoving frame is
$V_{\text{matter}} = \sqrt{2} \cdot G_F \cdot n_e(x),$
here we only consider electron's matter effect since in the astrophysical systems of interest, electron is the only lepton with a significant abundance.
We assume that the electron number density at the center of sun is $n_{e0} = 5.9 \times 10^{29}\ \text{m}^{-3}$ and the density is decaying with the form:

$$n_e(x) = n_{e0}\left(1 - \frac{x}{R_{\text{sun}}}\right).$$

The energy of the neutrinos are set to be $1 \times 10^6 eV$.



What if we want to study a system of neutrinos instead of just 1?

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Kinetic Theory

- BBGKY Formalism

Bogoliubov–Born–Green–Kirkwood–Yvon hierarchy (BBGKY) describes the dynamic statistics for large system of interacting particles through their distribution.

$$p^\mu \frac{\partial F}{\partial x^\mu} + \frac{dp^i}{d\tau}\frac{\partial F}{\partial p^i} = -p^\mu u_\mu S + i p^\mu n_\mu [H, F]$$

S is the collision term, u_μ and n_μ are 4-velocity and 4-normal vector, respectively. Under the assumption, the neutrino self-interaction is treated as an interaction between each neutrino and their mean-field neutrino medium in its vicinity.

Similar as the quantum mechanical system in the Heisenberg picture, where the density operator F evolves as:

$$\frac{dF}{dt} = -i[H, F]$$

The Hamiltonian operator is often decomposed as

$$H = H_{\text{vacuum}} + H_{\text{matter}} + H_{\text{neutrino}} \cdot$$

Assuming non-relativistic, isotropy, no drift and force terms.

Kinetic Theory

- Hamiltonians

The vaccum Hamiltonian

$$H_{\text{vacuum}} = U H^{(m)}_{\text{vacuum}} U^{\dagger},$$

in which

$$H^{(m)}_{\text{vacuum}} = \frac{1}{2E}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta m^2_{21} & 0 \\ 0 & 0 & \Delta m^2_{31} \end{pmatrix},$$

is the Hamiltonian in the neutrino mass basis, the unitary matrix U describes the mixing between the flavor and mass bases. Here we use the most commonly used Pontecorvo-Maki-NakagawaSakata (PMNS) matrix, which is defined as

$$U = U_{\text{PMNS}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix} \begin{bmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{\text{CP}}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{\text{CP}}} & 0 & c_{13} \end{bmatrix} \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$3 \times 3 - 4 - 1 = 4\ d.o.f. = $ 3 mixing angle + 1 CP phase.

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Kinetic Theory

- Hamiltonians

Charged-current interaction

$$H_{CC} = \frac{G_F}{\sqrt{2}} \int d^3\vec{x} \left[\bar{\phi}_e \gamma_\mu (1-\gamma_5) \phi_{\nu_e}\right], \left[\bar{\phi}_{\nu_e} \gamma^\mu (1-\gamma_5) \phi_e\right]$$

The mean field

$$\Gamma_{\nu_e}(\rho_e) = \frac{G_F}{\sqrt{2}} \sum_{h_e,h_e'} \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \int \frac{d^3\vec{p}\,'}{(2\pi)^3 2E_{p'}}$$

$$(2\pi)^3 \delta^3\left(\vec{p} + \vec{k} - \vec{p}\,' - \vec{k}\,'\right)$$

$$\left[\bar{u}_{\nu_e}\left(\vec{k}, h_{\nu_e}\right) \gamma_\mu (1-\gamma_5) u_{\nu_e}\left(\vec{k}\,', h_{\nu_e}'\right)\right]$$

$$\left[\bar{u}_e\left(\vec{p}, h_e\right) \gamma^\mu (1-\gamma_5) u_e\left(\vec{p}\,', h_e'\right)\right]$$

$$\left\langle a_e^\dagger(\vec{p}, h) a_e\left(\vec{p}\,', h'\right)\right\rangle.$$

Kinetic Theory

- Hamiltonians

Neutral-current interaction

$$H_{NC} = \frac{G_F}{2\sqrt{2}} \int d^3\vec{x} \left[\bar{\phi}_{\nu_e}\gamma_\mu(1-\gamma_5)\phi_{\nu_e}\right], \left[\bar{\phi}_{\nu_y}\gamma^\mu(1-\gamma_5)\phi_{\nu_y}\right]$$

with $\nu_y = \nu_e, \nu_\mu$ or ν_τ. The mean-field $\Gamma_{\nu_\alpha,\nu_\beta}(\rho_\nu)$ with $\nu_\alpha, \nu_\beta = \nu_e, \nu_\mu, \nu_\tau$

$$\Gamma_{\nu_\alpha,\nu_\beta}(\rho_\nu) = \frac{G_F}{2\sqrt{2}} \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \int \frac{d^3\vec{p}^{\,'}}{(2\pi)^3 2E_{p'}}$$

$$(2\pi)^3\delta^3\left(\vec{p} + \vec{k} - \vec{p}^{\,'} - \vec{k}^{\,'}\right)$$

$$\left[\bar{u}_{\nu_\beta}\left(\vec{k}, h_\beta\right)\gamma_\mu(1-\gamma_5)u_{\nu_\alpha}\left(\vec{k}^{\,'}, h'_\alpha\right)\right]$$

$$\left[\bar{u}_{\nu_\alpha}\left(\vec{p}, h_\alpha\right)\gamma^\mu(1-\gamma_5)u_{\nu_\beta}\left(\vec{p}^{\,'}, h'_\beta\right)\right]$$

$$\left\langle a^\dagger_{\nu_\alpha}\left(\vec{p}, h_\alpha\right)a_{\nu_\beta}\left(\vec{p}^{\,'}, h'_\beta\right)\right\rangle$$

Hamiltonian

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Kinetic Theory

- Examples

Analytical:

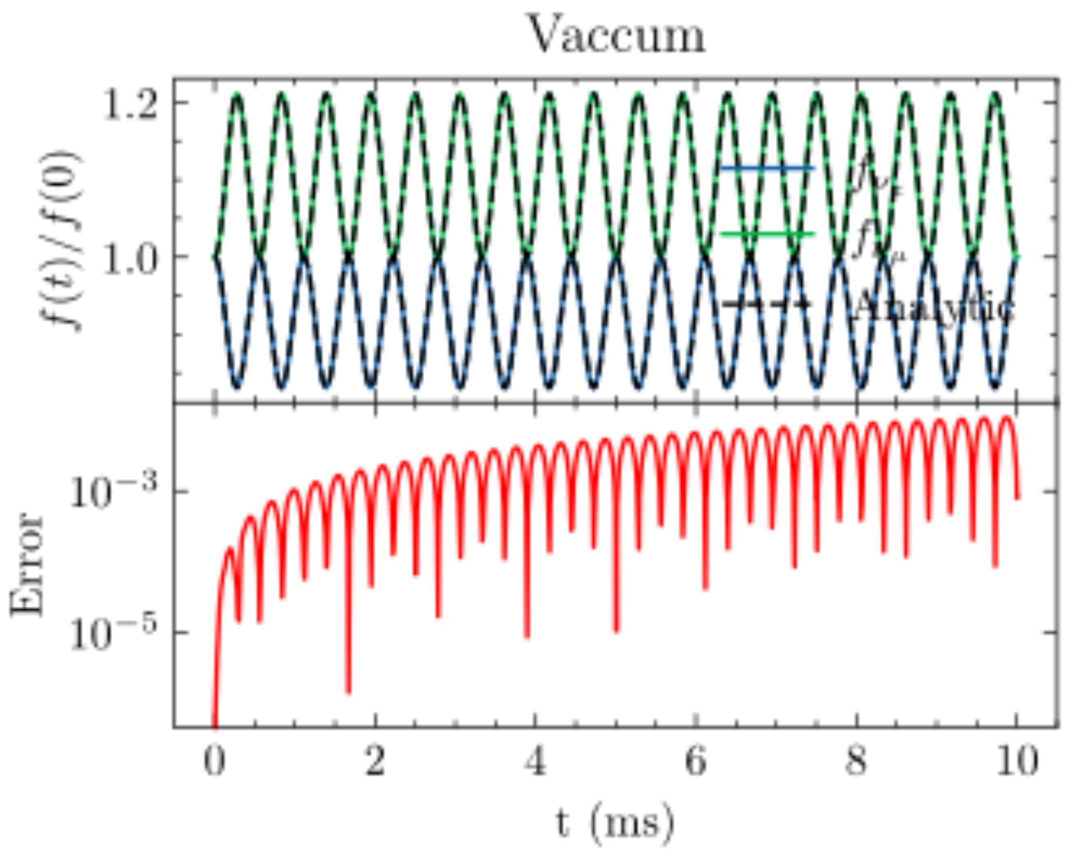
$$P_T(t) = \sin^2(2\theta_{12})\sin^2\left(\frac{c^4\Delta m_{12}^2 t}{4E\hbar}\right)$$

Thus, we expect the distribution function values to follow

$$\begin{aligned} f_{ee}(t) &= \left[1 - P_T(t)\right] f_{ee}(0) + P_T(t) f_{\mu\mu}(0) \\ f_{\mu\mu}(t) &= \left[1 - P_T(t)\right] f_{\mu\mu}(0) + P_T(t) f_{ee}(0) \end{aligned}$$

Input:

- $E = 5 \times 10^6$ eV
- $\theta_{12} = \pi/6$
- Two flavors with $\Delta m_{12}^2 = 7.53 \times 10^{-5}$ eV



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Kinetic Theory

- Examples

Analytical:
Mixing angle and mass squared difference are replaced by effective values of

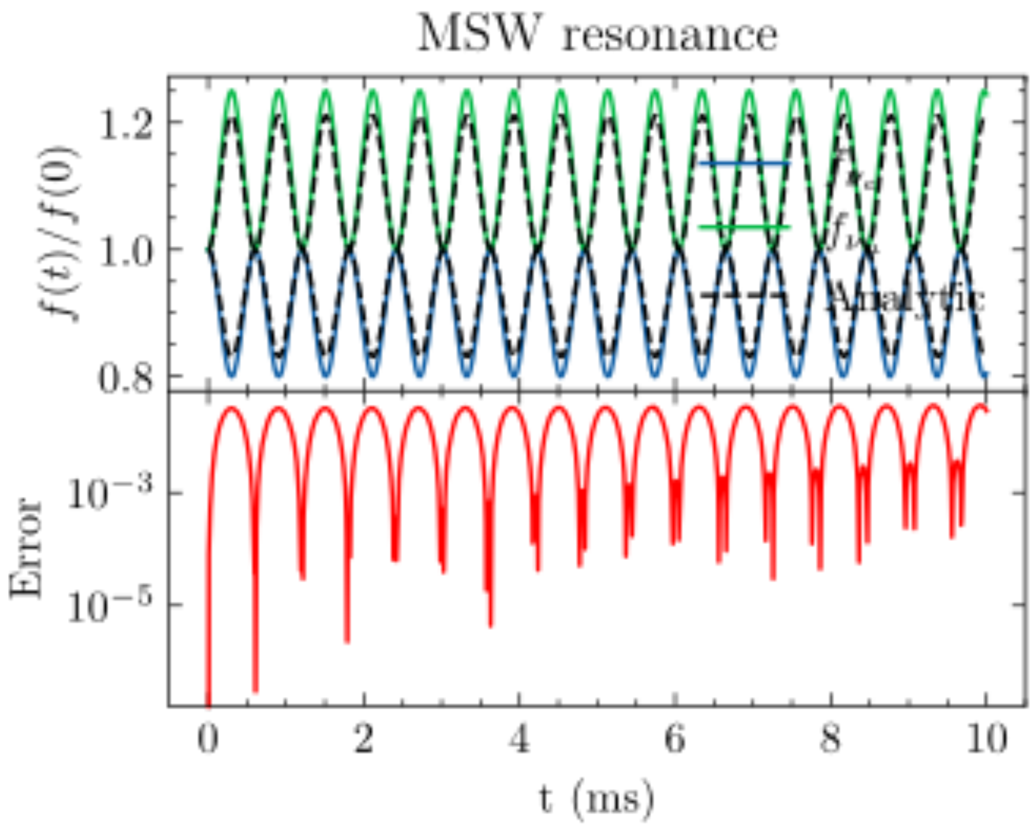
$$\sin^2\left(2\tilde{\theta}_{12}\right) = \frac{\sin^2(2\theta_{12})}{\sin^2(2\theta_{12}) + C^2}$$

$$\Delta\tilde{m}_{12}^2 = \Delta m_{12}^2\sqrt{\sin^2(2\theta_{12}) + C^2}$$

$$C = \cos(2\theta_{12}) - \frac{2VE}{\Delta m_{12}^2 c^4}$$

where $V = \sqrt{2}G_F\hbar^3c^3n_e$ is the matter potential. We assume Mikheyev-Smirnov-Wolfenstein (MSW) resonance ($C = 0$).
Input:

- $E = 5 \times 10^6$ eV
- Two flavors with $\Delta m_{12}^2 = 7.53 \times 10^{-5}$ eV



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Kinetic Theory

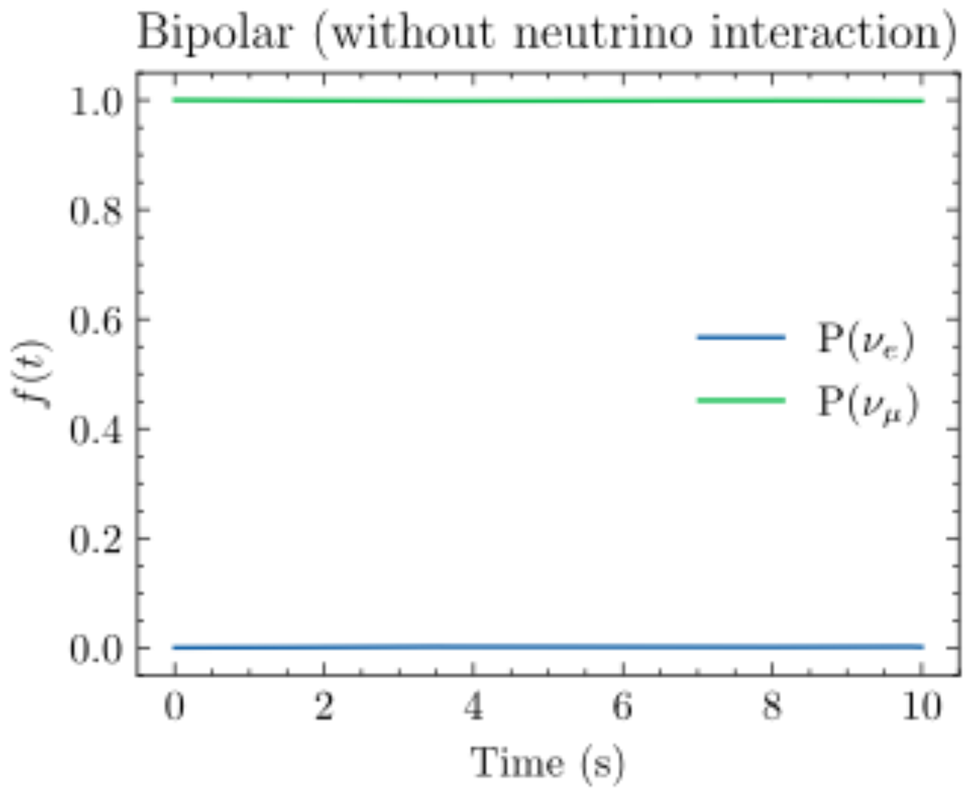
- Examples

Bipolar oscillations occur for pure electron anti/neutrinos and for pure anti/muon neutrinos. If we assume the distribution of anti/neutrinos are isotropic

$H_{\text{neutrino}} = 0$

Input:

- $E = 5.64 \times 10^{10}$ eV
- Two flavors with $\Delta m_{12}^2 = 7.53 \times 10^{-5}$ eV
- Initial distribution: $f_{\mu\mu} = \bar{f}_{\mu\mu} = 1$
- Background matter density $= 0$
- $\theta_{12} = 0.01$



Kinetic Theory

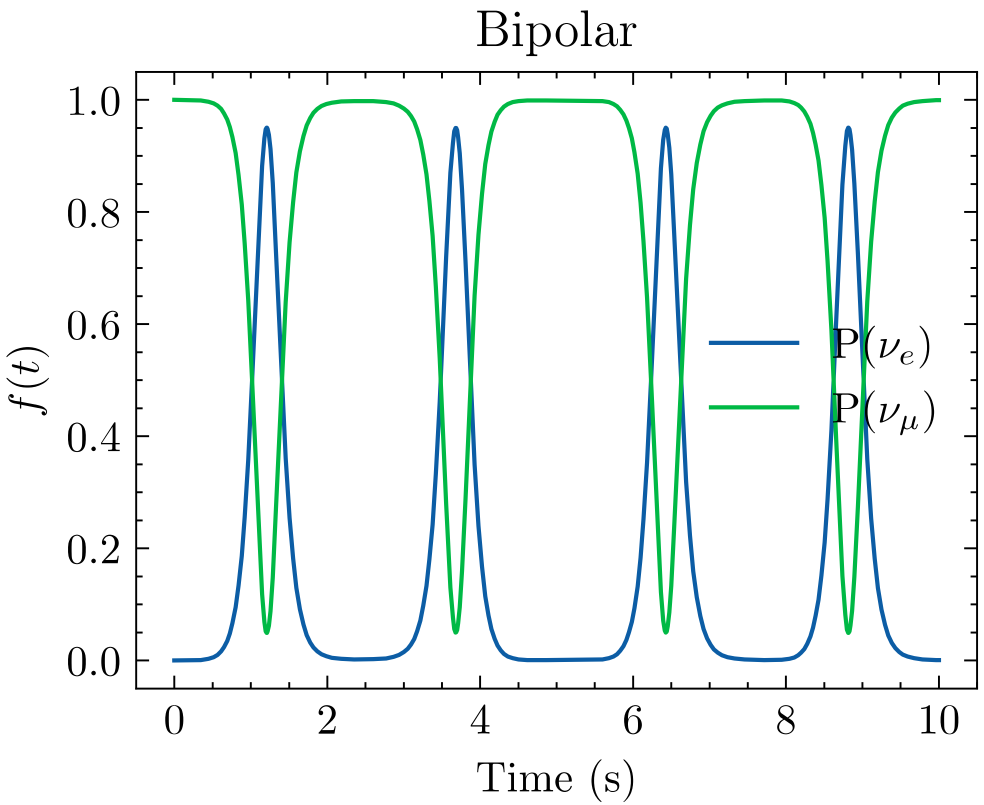
- Examples

Bipolar oscillations occur for pure electron anti/neutrinos and for pure anti/muon neutrinos. If we assume the distribution of anti/neutrinos are isotropic

$$H_{\mathrm{neutrino}} = \frac{\Delta m_{12}^2 c^4}{2E}(f - \bar{f}^*)\,.$$

Input:

- $E = 5.64 \times 10^{10}$ eV
- Two flavors with $\Delta m_{12}^2 = 7.53 \times 10^{-5}$ eV
- Initial distribution: $f_{\mu\mu} = \bar{f}_{\mu\mu} = 1$
- Background matter density $= 0$
- $\theta_{12} = 0.01$



Numerical Setup

- Angular Discretization

To study the neutrino propagation in a space with anisotropic collision, we need angular discretization, which is usually implemented on the distribution function to separate the radial and angular parts. For example, in our case, we can simply consider

$$F\left(t, x^i, \Omega\right) = \sum_{A=0}^{N-1} F^A\left(t, x^i\right)\Psi_A(\Omega) := F^A\Psi_A,$$

where the F would be matrix element of the density matrix we defined previously. Thus, the equation we have would be:

$$\Psi_A \frac{\partial F^A}{\partial t} + \Omega^i \Psi_A \frac{\partial F^A}{\partial r^i} = -i\left[H^A\Psi_A, F^A\Psi_A\right].$$

$$\Psi_A \frac{\partial F^A}{\partial t} + \Omega^i \Psi_A \frac{\partial F^A}{\partial x^i} = \mathbb{C}[F],$$

Multiplying by $\Psi^B = \Psi_B$ and integrating over $\mathbb{S}_2$ with respect to $d\Omega$, we obtain

$$M_A^B \frac{\partial F^A}{\partial t} + S_A^{iB} \frac{\partial F^A}{\partial x^i} = \mathbb{S}^B[F],$$

where

$$M_A^B = \int_{\mathbb{S}_2} \Psi^B \Psi_A d\Omega,$$

$$S_A^{iB} = \int_{\mathbb{S}_2} \Omega^i \Psi^B \Psi_A d\Omega, \ \mathbb{S}^B[F] = \int_{\mathbb{S}_2} \Psi^B \mathbb{C}[F] d\Omega.$$

are the mass, stiffness and source matrices respectively.

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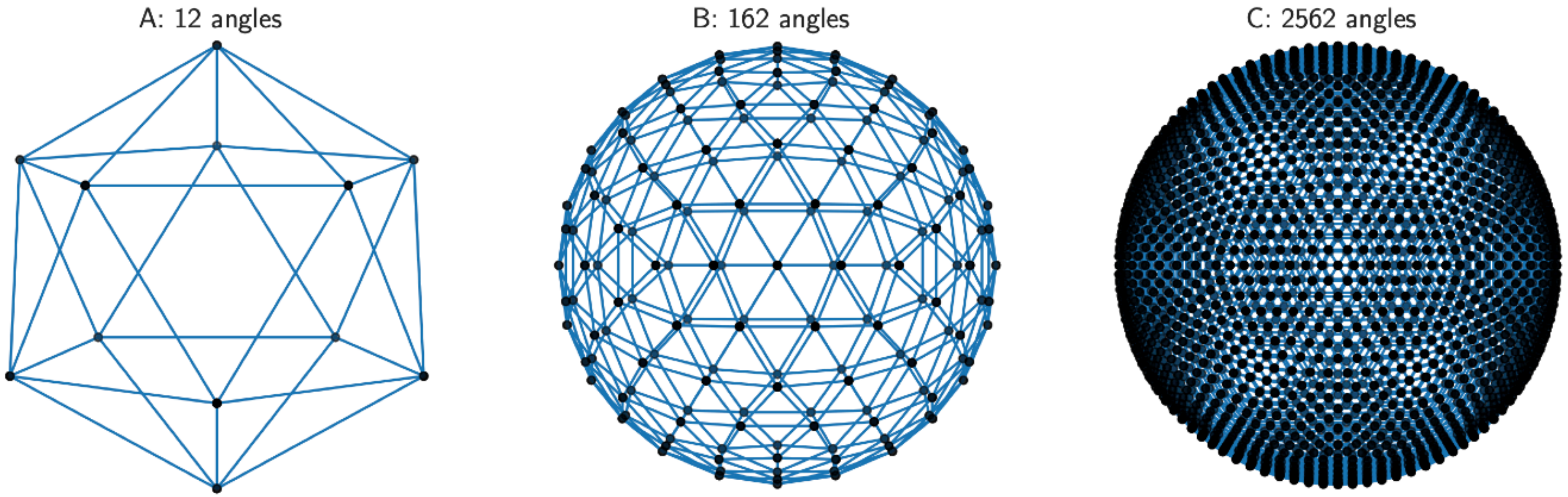
Numerical Setup

- Angular Discretization



Credit: Bhattacharyya (2022)

The simplest geodesic grid consists of 12 angular points which are vertices of a regular icosahedron, the Cartesian coordinates of which are given by

$$\frac{1}{\sqrt{1+\varphi^2}}(0, \pm 1, \pm \varphi),\ \frac{1}{\sqrt{1+\varphi^2}}(\pm 1, \pm \varphi, 0),\ \frac{1}{\sqrt{1+\varphi^2}}(\pm \varphi, 0, \pm 1),\ \varphi = \frac{1+\sqrt{5}}{2}.$$

Here φ is the golden radio.

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Numerical Setup

- Angular Discretization

We have firstly:

$$H_{\text{vacuum}} = \begin{pmatrix} H_{v_1} & H_{v_2} & H_{v_3} \\ H_{v_4} & H_{v_5} & H_{v_6} \\ H_{v_7} & H_{v_8} & H_{v_9} \end{pmatrix}$$

The values of $H_{v_{1-9}}$ can be either taking from theoretical assumptions or calculated by experimentally measured values of U_{PMNS} .

Given also the Hamiltonian for matter.

$$H_{\text{matter}} = \sqrt{2}G_F \begin{pmatrix} n_e - n_{\bar{e}} & 0 & 0 \\ 0 & n_\mu - n_{\bar{\mu}} & 0 \\ 0 & 0 & n_\tau - n_{\bar{\tau}} \end{pmatrix},$$

Note that they are just leading order terms from QFT calculation. Also note that here we assume the distribution of charged lepton and antilepton is isotropic, hence there is no angle dependence included.

And the neutrino self-interaction Hamiltonian:

$$H_{\text{neutrino}} = \sqrt{2}G_F(1 - \cos\theta_A) \times$$

$$\times \Psi_A \begin{pmatrix} \left(f^A_{\nu_e} - \bar{f}^{*A}_{\bar{\nu}}\right) & \left(f^A_{e\mu} - \bar{f}^{*A}_{e\mu}\right) & \left(f^A_{e\tau} - \bar{f}^{*A}_{\bar{e}_e}\right) \\ \left(f^A_{e\mu} - \bar{f}^{*A}_{e\mu}\right)^* & \left(f^A_{\nu_\mu} - \bar{f}^{*A}_{\bar{\nu}_\mu}\right) & \left(f^A_{\mu\tau} - \bar{f}^{*A}_{\mu\tau}\right) \\ \left(f^A_{e\tau} - \bar{f}^{*A}_{e\tau}\right)^* & \left(f^A_{\mu\tau} - \bar{f}^{*A}_{\mu\tau}\right)^* & \left(f^A_{\nu_\tau} - \bar{f}^{*A}_{\bar{\nu}_\tau}\right) \end{pmatrix}.$$

where the f^A_x are the matrix element of the angular discretized density matrix F^A.

Summary

- Neutrino Oscillations
- Quantum Kinetics Theory
- Numerical Scheme

Questions welcome

Thank you for your attentions!

Acknowledgement:

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Supplementary materials

$$\rho_\nu = \begin{pmatrix} \langle a^\dagger_{\nu_\alpha,i} a_{\nu_\alpha,i} \rangle & \langle a^\dagger_{\nu_\beta,j} a_{\nu_\alpha,i} \rangle & \langle a^\dagger_{\nu_\gamma,k} a_{\nu_\alpha,i} \rangle \\ \langle a^\dagger_{\nu_\alpha,i} a_{\nu_\beta,j} \rangle & \langle a^\dagger_{\nu_\beta,j} a_{\nu_\beta,j} \rangle & \langle a^\dagger_{\nu_\gamma,k} a_{\nu_\beta,j} \rangle \\ \langle a^\dagger_{\nu_\alpha,i} a_{\nu_\gamma,k} \rangle & \langle a^\dagger_{\nu_\beta,j} a_{\nu_\gamma,k} \rangle & \langle a^\dagger_{\nu_\gamma,k} a_{\nu_\gamma,k} \rangle \end{pmatrix}.$$

$$\phi(\vec{x}) = \sum_h \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \left[a(\vec{p}, h) u_{\vec{p},h} e^{i\vec{p}\cdot\vec{x}} + b^\dagger(\vec{p}, h) v_{\vec{p},h} e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$G_F \equiv \frac{g^2}{4\sqrt{2} M_W^2}$$

in which g is the weak coupling constant, and M_W is the mass of the W bosons.

The neutrinos and antineutrinos are related by CP symmetry, such that we have the following transformation rule:

$$\nu_\alpha \xleftrightarrow{\text{CP}} \overline{\nu}_\alpha,$$

When we consider the process of time evolution, we have:

$$\nu_\alpha \to \nu_\beta \xleftrightarrow{\text{CP}} \overline{\nu}_\alpha \to \overline{\nu}_\beta,$$

Weak Interaction

In a charged-current weak scattering process, a charged weak interaction involves the exchange of a W^+, W^-bosons. For example, when a neutrino interacts with a nucleus in a neutrino detector, it can scatter off an atomic nucleus via a charged-current weak interaction, producing a charged lepton (an electron, muon or tau) and changing the quark flavor. The reaction can be represented as follows:

$\nu + \mathrm{N} \rightarrow \mathrm{I} + \mathrm{X}$,

where ν is the neutrino, N is the atomic nucleus, I is the charged lepton, and X is the recoiling nucleus.

In a neutral-current weak scattering process, the exchange of a Z boson is involved. For example, when a neutrino interacts with an atomic nucleus in a neutrino detector, it can scatter off the nucleus via a neutral-current weak interaction, producing a recoiling nucleus and a neutrino of the same flavor. The reaction can be represented as follows:

$\nu + \mathrm{N} \rightarrow \nu + \mathrm{X}$

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Mean Field Approximation

Writing such an equation more explicitly, it reads

$i\dot{\rho}_{1,ij} - \left[H_0(1) + \Gamma_1(\rho), \rho_1\right]_{ij} = 0$

with

$\Gamma_{1,ij}(\rho) = \sum_{mn} v_{(im,jn)}\rho_{2,nm}\,.$

The mean-field potential is built up from a complete set of one-body density matrix components for particle $2\rho_{2,nm}$, each contributing with the matrix element $v_{(im,jn)} = \left\langle im \middle| V_{12} \middle| jn \right\rangle$, with incoming (outgoing) single-particle states.