Various approximations around the Quantum Kinetic Equations

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Outline

1. The Quantum Kinetic Equations

2. Neutrino evolution in the early Universe: the *Adiabatic Transfer of Averaged Oscillations* approximation

1. The Quantum Kinetic Equations

Introducing the QKEs

- In order to describe the evolution of a statistical ensemble of neutrinos: combination of kinetic theory and quantum mechanics.

Boltzmann equation

Flavor mixing

- Generalization of the distribution functions: the **“density matrix”**

$$\begin{pmatrix} f_{\nu_e} & & \\ & f_{\nu_\mu} & \\ & & f_{\nu_\tau} \end{pmatrix} \longrightarrow \begin{pmatrix} \varrho_{ee} & \varrho_{e\mu} & \varrho_{e\tau} \\ \varrho_{\mu e} & \varrho_{\mu\mu} & \varrho_{\mu\tau} \\ \varrho_{\tau e} & \varrho_{\tau\mu} & \varrho_{\tau\tau} \end{pmatrix}$$

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- Evolution equation: the **Quantum Kinetic Equation**

$$\boxed{i \frac{\mathrm{d}\varrho(\vec{x}, \vec{p}, t)}{\mathrm{d}t} = [\mathcal{H}_{\mathrm{vac}} + \mathcal{H}_{\mathrm{mat}} + \mathcal{H}_{\mathrm{self}}, \varrho] + i\,\mathcal{I}(\varrho, \bar{\varrho})}$$

Derivation of the QKE

Various approaches to derive the Quantum Kinetic Equation:

- Perturbative expansion [G. Sigl, G. Raffelt, *Nucl. Phys. B* 406, 423 (1993)]
- Close Time Path formalism [D. Blaschke, V. Cirigliano, *Phys. Rev. D* 94, 033009 (2016)]
- Extended BBGKY hierarchy [**JF**, C. Pitrou, M.C. Volpe, *JCAP* 12, 015 (2020)]

Central object: *s*-body reduced density matrix

$$\varrho^{i_1\cdots i_s}_{j_1\cdots j_s} \equiv \langle \hat{a}^\dagger_{j_s}\cdots\hat{a}^\dagger_{j_1}\hat{a}_{i_1}\cdots\hat{a}_{i_s}\rangle$$

In particular, one-body density matrix $\varrho^i_j \equiv \langle \hat{a}^\dagger_j \hat{a}_i\rangle$

$$\left(\varrho^{\phi_i(\vec{p}_i,h_i)}_{\phi_j(\vec{p}_j,h_j)} = \langle \hat{a}^\dagger_{\phi_j}(\vec{p}_j,h_j)\,\hat{a}_{\underline{\phi_i}}(\underline{\vec{p}_i},\underline{h_i})\rangle\right)$$

species, momentum, helicity

Extended BBGKY formalism

Hierarchy of equations

- Hamiltonian (second quantization)

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} = \sum_{i,j} t^i_j \, \hat{a}^\dagger_i \hat{a}_j + \frac{1}{4} \sum_{i,j,k,l} \tilde{v}^{ik}_{jl} \, \hat{a}^\dagger_i \hat{a}^\dagger_k \hat{a}_l \hat{a}_j$$

[C. Volpe et al., *Phys. Rev. D* 87 (2013)]

Kinetic term

Two-body interactions

- BBGKY hierarchy

Ehrenfest theorem

$$\boxed{i \frac{\mathrm{d}\langle \hat{a}^\dagger_j \hat{a}_i \rangle}{\mathrm{d}t} = \langle [\hat{a}^\dagger_j \hat{a}_i, \hat{H}] \rangle}$$

$$\begin{cases} i \dfrac{\mathrm{d}\varrho^i_j}{\mathrm{d}t} = \left(t^i_k \varrho^k_j - \varrho^i_k t^k_j\right) + \dfrac{1}{2}\left(\tilde{v}^{ik}_{ml} \varrho^{ml}_{jk} - \varrho^{ik}_{ml} \tilde{v}^{ml}_{jk}\right) \\ i \dfrac{\mathrm{d}\varrho^{ik}_{jl}}{\mathrm{d}t} = \left(t^i_r \varrho^{rk}_{jl} + t^k_p \varrho^{ip}_{jl} + \dfrac{1}{2} \tilde{v}^{ik}_{rp} \varrho^{rp}_{jl} - \varrho^{ik}_{rl} t^r_j - \varrho^{ik}_{jp} t^p_l - \dfrac{1}{2} \varrho^{ik}_{rp} \tilde{v}^{rp}_{jl}\right) \\ \qquad\qquad + \dfrac{1}{2}\left(\tilde{v}^{im}_{rn} \varrho^{rkn}_{jlm} + \tilde{v}^{km}_{pn} \varrho^{ipn}_{jlm} - \varrho^{ikm}_{rln} \tilde{v}^{rn}_{jm} - \varrho^{ikm}_{jpn} \tilde{v}^{pn}_{lm}\right) \end{cases}$$

1-body density matrix

2-body density matrix

3-body density matrix

Extended BBGKY formalism

Hierarchy of equations

- Hamiltonian (second quantization)

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} = \sum_{i,j} t^i_j \, \hat{a}^\dagger_i \hat{a}_j + \frac{1}{4} \sum_{i,j,k,l} \tilde{v}^{ik}_{jl} \, \hat{a}^\dagger_i \hat{a}^\dagger_k \hat{a}_l \hat{a}_j$$

[C. Volpe et al., *Phys. Rev. D* 87 (2013)]

Kinetic term

Two-body interactions

- BBGKY hierarchy

Need to truncate this hierarchy

theorem

$$\boxed{i\frac{\langle \quad_j \quad_i\rangle}{\mathrm{d}t} = \langle[\hat{a}^\dagger_j \hat{a}_i, \hat{H}]\rangle}$$

$$\begin{cases} i\dfrac{\mathrm{d}\varrho^i_j}{\mathrm{d}t} = \left(t^i_k \varrho^k_j - \varrho^i_k t^k_j\right) + \dfrac{1}{2}\left(\tilde{v}^{ik}_{ml} \varrho^{ml}_{jk} - \varrho^{ik}_{ml} \tilde{v}^{ml}_{jk}\right) \\ i\dfrac{\mathrm{d}\varrho^{ik}_{jl}}{\mathrm{d}t} = \left(t^i_r \varrho^{rk}_{jl} + t^k_p \varrho^{ip}_{jl} + \dfrac{1}{2}\tilde{v}^{ik}_{rp} \varrho^{rp}_{jl} - \varrho^{ik}_{rl} t^r_j - \varrho^{ik}_{jp} t^p_l - \dfrac{1}{2}\varrho^{ik}_{rp} \tilde{v}^{rp}_{jl}\right) \\ \qquad\qquad + \dfrac{1}{2}\left(\tilde{v}^{im}_{rn} \varrho^{rkn}_{jlm} + \tilde{v}^{km}_{pn} \varrho^{ipn}_{jlm} - \varrho^{ikm}_{rln} \tilde{v}^{rn}_{jm} - \varrho^{ikm}_{jpn} \tilde{v}^{pn}_{lm}\right) \end{cases}$$

1-body density matrix

2-body density matrix

3-body density matrix

Extended BBGKY formalism

Mean-field truncation

- Correlated and uncorrelated contributions

$$\varrho^{ik}_{jl} \equiv 2\varrho^{i}_{[j}\varrho^{k}_{l]} + C^{ik}_{jl} \equiv \varrho^{i}_{j}\varrho^{k}_{l} - \varrho^{i}_{l}\varrho^{k}_{j} + C^{ik}_{jl}$$

Extended BBGKY formalism

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$$\Longrightarrow \; i\frac{\mathrm{d}\varrho^{i}_{j}}{\mathrm{d}t} = \left(\left[t^{i}_{k} + \Gamma^{i}_{k}\right]\varrho^{k}_{j} - \varrho^{i}_{k}\left[t^{k}_{j} + \Gamma^{k}_{j}\right]\right) + \frac{1}{2}\left(\tilde{v}^{ik}_{ml}C^{ml}_{jk} - C^{ik}_{ml}\tilde{v}^{ml}_{jk}\right)$$

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Mean-field potential $\boxed{\Gamma^{i}_{j} = \sum_{k,l}\tilde{v}^{ik}_{jl}\varrho^{l}_{k}}$

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Mean-field potential $\boxed{\Gamma^{i}_{j} = \sum_{k,l}\tilde{v}^{ik}_{jl}\varrho^{l}_{k}}$

- Simplest closure: Hartree-Fock (or *mean-field*) approximation

but need to account for correlations due to two-body collisions…

Extended BBGKY formalism

Mean-field truncation

- Correlated and uncorrelated contributions

$$\varrho^{ik}_{jl} \equiv 2\varrho^{i}_{[j}\varrho^{k}_{l]} + C^{ik}_{jl} \equiv \varrho^{i}_{j}\varrho^{k}_{l} - \varrho^{i}_{l}\varrho^{k}_{j} + \cancel{C^{ik}_{jl}}$$

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Mean-field potential

$$\boxed{\Gamma^{i}_{j} = \sum_{k,l}\tilde{v}^{ik}_{jl}\,\varrho^{l}_{k}}$$

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Molecular chaos approximation

Extended BBGKY formalism

Towards the collision term

- *Molecular chaos* assumption = correlations are built from collisions between uncorrelated particles

$$i\frac{\mathrm{d}C^{ik}_{jl}}{\mathrm{d}t} \simeq \left[t^i_r C^{rk}_{jl} + t^k_p C^{ip}_{jl} - C^{ik}_{rl} t^r_j - C^{ik}_{jp} t^p_l\right] \\ + (\hat{1}-\varrho)^i_r (\hat{1}-\varrho)^k_p \, \tilde{v}^{rp}_{sq} \, \varrho^s_j \varrho^q_l - \varrho^i_r \varrho^k_p \, \tilde{v}^{rp}_{sq} \, (\hat{1}-\varrho)^s_j (\hat{1}-\varrho)^q_l$$

Pauli-blocking factors

Extended BBGKY formalism

Towards the collision term

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Pauli-blocking factors

- Evolution equation

$$i\frac{\mathrm{d}\varrho^i_j}{\mathrm{d}t} = \left(\left[t^i_k + \Gamma^i_k\right] \varrho^k_j - \varrho^i_k \left[t^k_j + \Gamma^k_j\right]\right) + \frac{1}{2}\left(\tilde{v}^{ik}_{ml} C^{ml}_{jk} - C^{ik}_{ml} \tilde{v}^{ml}_{jk}\right)$$

$$= \left[\hat{t} + \hat{\Gamma}, \hat{\varrho}\right]^i_j + i\,\hat{\mathcal{C}}^i_j$$

Extended BBGKY formalism

Collision term

$$i\,\frac{\mathrm{d}\varrho^{i}_{j}}{\mathrm{d}t} = \left[\hat{t} + \hat{\Gamma}, \hat{\varrho}\right]^{i}_{j} + i\,\hat{\mathcal{C}}^{i}_{j}$$

$$\begin{aligned}\mathcal{C}^{i_1}_{i'_1} \propto \frac{1}{4}\Big(&\tilde{v}^{i_1 i_2}_{i_3 i_4}\,\varrho^{i_3}_{j_3}\,\varrho^{i_4}_{j_4}\,\tilde{v}^{j_3 j_4}_{j_1 j_2}\,(\hat{1}-\varrho)^{j_1}_{i'_1}\,(\hat{1}-\varrho)^{j_2}_{i_2} - \tilde{v}^{i_1 i_2}_{i_3 i_4}\,(\hat{1}-\varrho)^{i_3}_{j_3}\,(\hat{1}-\varrho)^{i_4}_{j_4}\,\tilde{v}^{j_3 j_4}_{j_1 j_2}\,\varrho^{j_1}_{i'_1}\,\varrho^{j_2}_{i_2}\\ &+(\hat{1}-\varrho)^{i_1}_{j_1}\,(\hat{1}-\varrho)^{i_2}_{j_2}\,\tilde{v}^{j_1 j_2}_{j_3 j_4}\,\varrho^{j_3}_{i_3}\,\varrho^{j_4}_{i_4}\,\tilde{v}^{i_3 i_4}_{i'_1 i_2} - \varrho^{i_1}_{j_1}\,\varrho^{i_2}_{j_2}\,\tilde{v}^{j_1 j_2}_{j_3 j_4}\,(\hat{1}-\varrho)^{j_3}_{i_3}\,(\hat{1}-\varrho)^{j_4}_{i_4}\,\tilde{v}^{i_3 i_4}_{i'_1 i_2}\Big)\end{aligned}$$

Extended BBGKY formalism

Collision term

$$i\,\frac{\mathrm{d}\varrho^{i}_{j}}{\mathrm{d}t} = \left[\hat{t} + \hat{\Gamma}, \hat{\varrho}\right]^{i}_{j} + i\,\hat{\mathcal{C}}^{i}_{j}$$

$$\begin{aligned}\mathcal{C}^{i_1}_{i'_1} \propto \frac{1}{4}\Big(& \tilde{v}^{i_1 i_2}_{i_3 i_4}\,\varrho^{i_3}_{j_3}\,\varrho^{i_4}_{j_4}\,\tilde{v}^{j_3 j_4}_{j_1 j_2}\,(\hat{1}-\varrho)^{j_1}_{i'_1}\,(\hat{1}-\varrho)^{j_2}_{i_2} - \tilde{v}^{i_1 i_2}_{i_3 i_4}\,(\hat{1}-\varrho)^{i_3}_{j_3}\,(\hat{1}-\varrho)^{i_4}_{j_4}\,\tilde{v}^{j_3 j_4}_{j_1 j_2}\,\varrho^{j_1}_{i'_1}\,\varrho^{j_2}_{i_2} \\ & + (\hat{1}-\varrho)^{i_1}_{j_1}\,(\hat{1}-\varrho)^{i_2}_{j_2}\,\tilde{v}^{j_1 j_2}_{j_3 j_4}\,\varrho^{j_3}_{i_3}\,\varrho^{j_4}_{i_4}\,\tilde{v}^{i_3 i_4}_{i'_1 i_2} - \varrho^{i_1}_{j_1}\,\varrho^{i_2}_{j_2}\,\tilde{v}^{j_1 j_2}_{j_3 j_4}\,(\hat{1}-\varrho)^{j_3}_{i_3}\,(\hat{1}-\varrho)^{j_4}_{i_4}\,\tilde{v}^{i_3 i_4}_{i'_1 i_2}\Big)\end{aligned}$$

Gain **Loss**

Extended BBGKY formalism

Collision term

$$i\frac{\mathrm{d}\varrho^i_j}{\mathrm{d}t} = \left[\hat{t}+\hat{\Gamma},\,\hat{\varrho}\right]^i_j + i\,\hat{\mathcal{C}}^i_j$$

$$\begin{aligned}\mathcal{C}^{i_1}_{i_1'} \propto \frac{1}{4}\Big(&\tilde{v}^{i_1 i_2}_{i_3 i_4}\,\boxed{f_3\, f_4}\,\tilde{v}^{j_3 j_4}_{j_1 j_2}\,\boxed{(1-f_1)(1-f_2)} - \tilde{v}^{i_1 i_2}_{i_3 i_4}(\hat{1}-\varrho)^{i_3}_{j_3}(\hat{1}-\varrho)^{i_4}_{j_4}\,\tilde{v}^{j_3 j_4}_{j_1 j_2}\,\varrho^{j_1}_{i_1'}\,\varrho^{j_2}_{i_2} \\ &+(\hat{1}-\varrho)^{i_1}_{j_1}(\hat{1}-\varrho)^{i_2}_{j_2}\,\tilde{v}^{j_1 j_2}_{j_3 j_4}\,\varrho^{j_3}_{i_3}\,\varrho^{j_4}_{i_4}\,\tilde{v}^{i_3 i_4}_{i_1' i_2} - \varrho^{i_1}_{j_1}\,\varrho^{i_2}_{j_2}\,\boxed{\tilde{v}^{j_1 j_2}_{j_3 j_4}}(\hat{1}-\varrho)^{j_3}_{i_3}(\hat{1}-\varrho)^{j_4}_{i_4}\,\boxed{\tilde{v}^{i_3 i_4}_{i_1' i_2}}\Big)\end{aligned}$$

$$\langle|\mathscr{M}|^2\rangle$$

Extended BBGKY formalism

Collision term

$$i\frac{\mathrm{d}\varrho^i_j}{\mathrm{d}t} = \left[\hat{t}+\hat{\Gamma},\hat{\varrho}\right]^i_j + i\,\hat{\mathcal{C}}^i_j$$

$$\begin{aligned}\mathcal{C}^{i_1}_{i'_1} \propto \frac{1}{4}\Big(&\tilde{v}^{i_1 i_2}_{i_3 i_4}\, f_3\, f_4\, \tilde{v}^{j_3 j_4}_{j_1 j_2}\,(1-f_1)(1-f_2) - \tilde{v}^{i_1 i_2}_{i_3 i_4}(\hat{1}-\varrho)^{i_3}_{j_3}(\hat{1}-\varrho)^{i_4}_{j_4}\,\tilde{v}^{j_3 j_4}_{j_1 j_2}\,\varrho^{j_1}_{i'_1}\,\varrho^{j_2}_{i_2}\\ &+(\hat{1}-\varrho)^{i_1}_{j_1}(\hat{1}-\varrho)^{i_2}_{j_2}\,\tilde{v}^{j_1 j_2}_{j_3 j_4}\,\varrho^{j_3}_{i_3}\,\varrho^{j_4}_{i_4}\,\tilde{v}^{i_3 i_4}_{i'_1 i_2} - \varrho^{i_1}_{j_1}\,\varrho^{i_2}_{j_2}\,\tilde{v}^{j_1 j_2}_{j_3 j_4}(\hat{1}-\varrho)^{j_3}_{i_3}(\hat{1}-\varrho)^{j_4}_{i_4}\,\tilde{v}^{i_3 i_4}_{i'_1 i_2}\Big)\end{aligned}$$

$\langle|\mathscr{M}|^2\rangle$

After computing the interaction matrix elements,

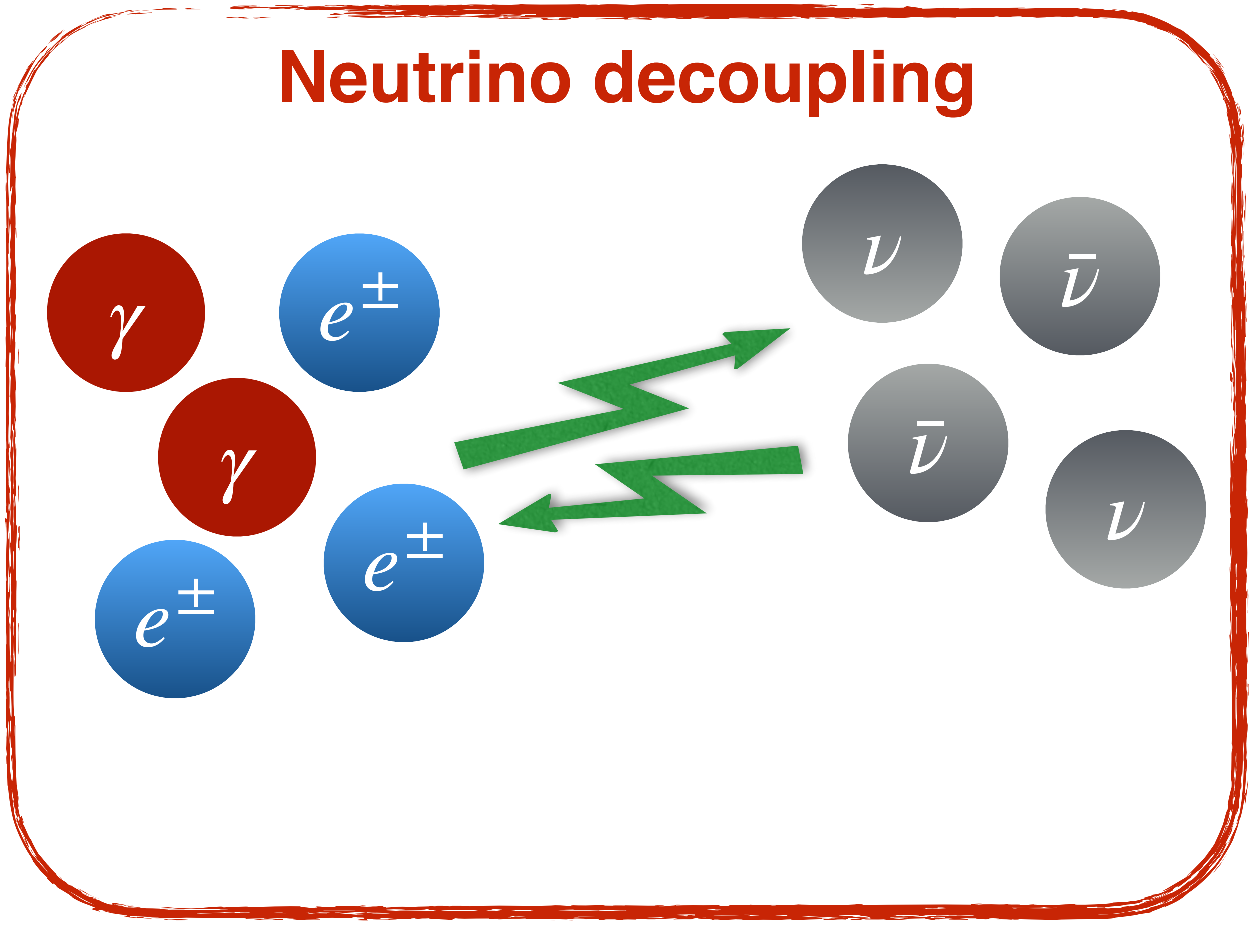
$$\boxed{i\frac{\mathrm{d}\varrho(\vec{x},\vec{p},t)}{\mathrm{d}t} = [\mathcal{H}_{\mathrm{vac}}+\mathcal{H}_{\mathrm{mat}}+\mathcal{H}_{\mathrm{self}},\varrho] + i\,\mathcal{I}(\varrho,\bar{\varrho})}$$

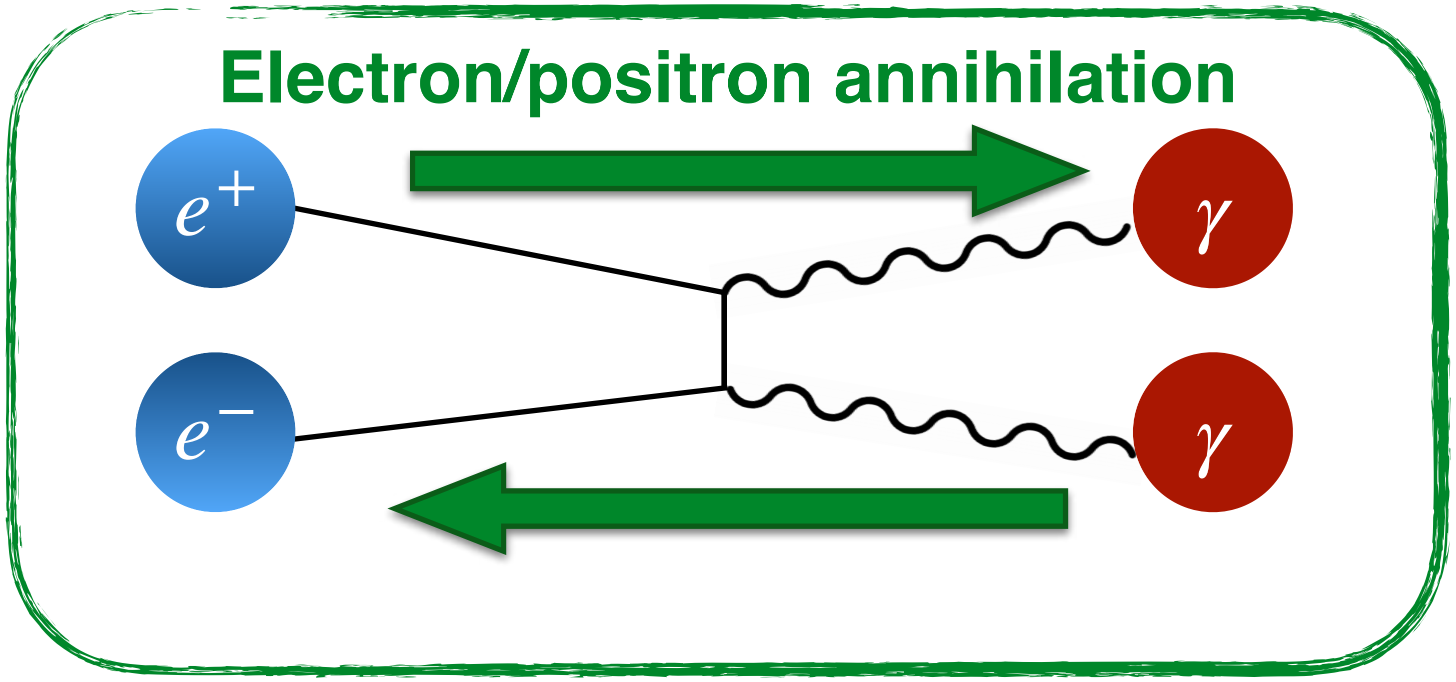
$\Longrightarrow$ The QKEs are an approximation neglecting higher than 2-body correlations.

2. Neutrino evolution in the early Universe

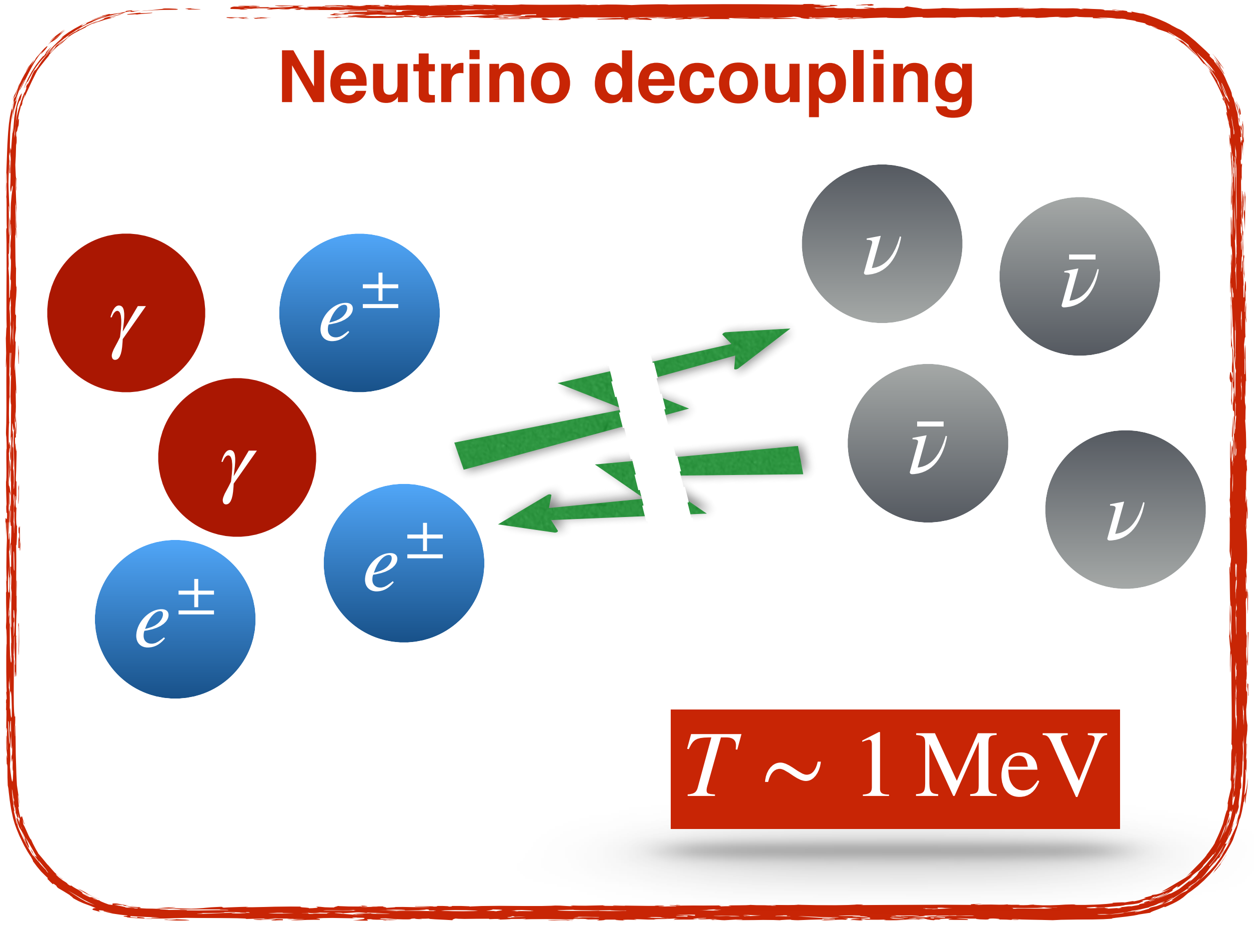
The *Adiabatic Transfer of Averaged Oscillations* approximation

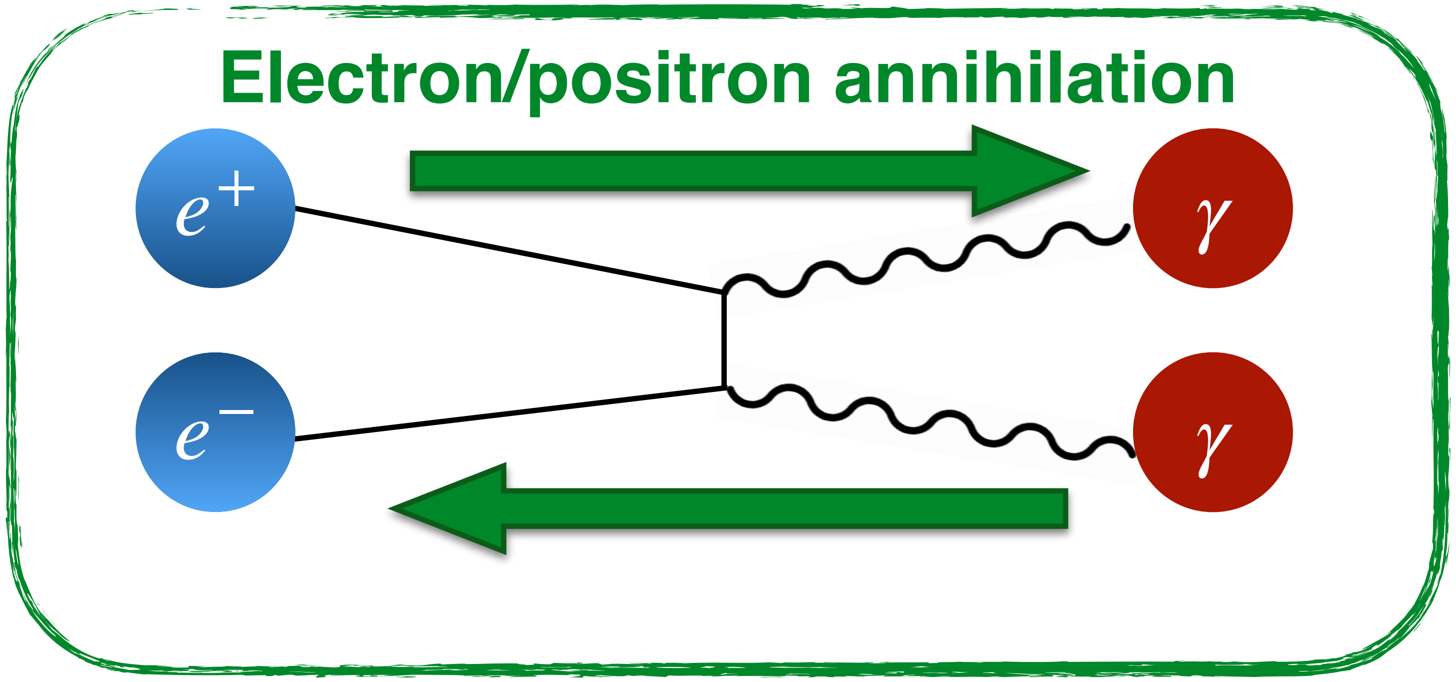
Neutrino decoupling


Neutrino decoupling


Electron/positron annihilation

Neutrino decoupling

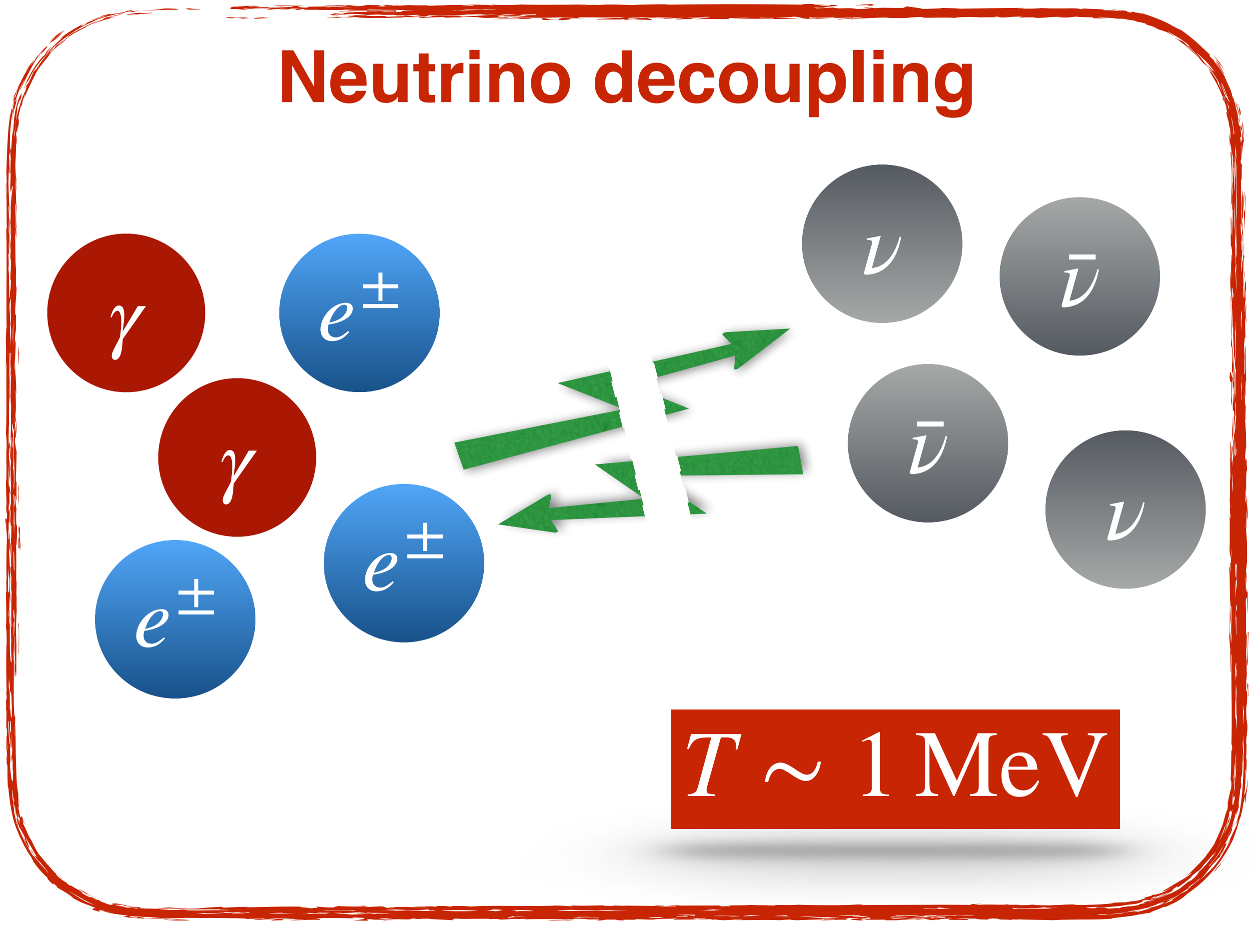




Below this temperature

$$T_{\nu_e} = T_{\nu_\mu} = T_{\nu_\tau} \propto a^{-1}$$

Neutrino decoupling

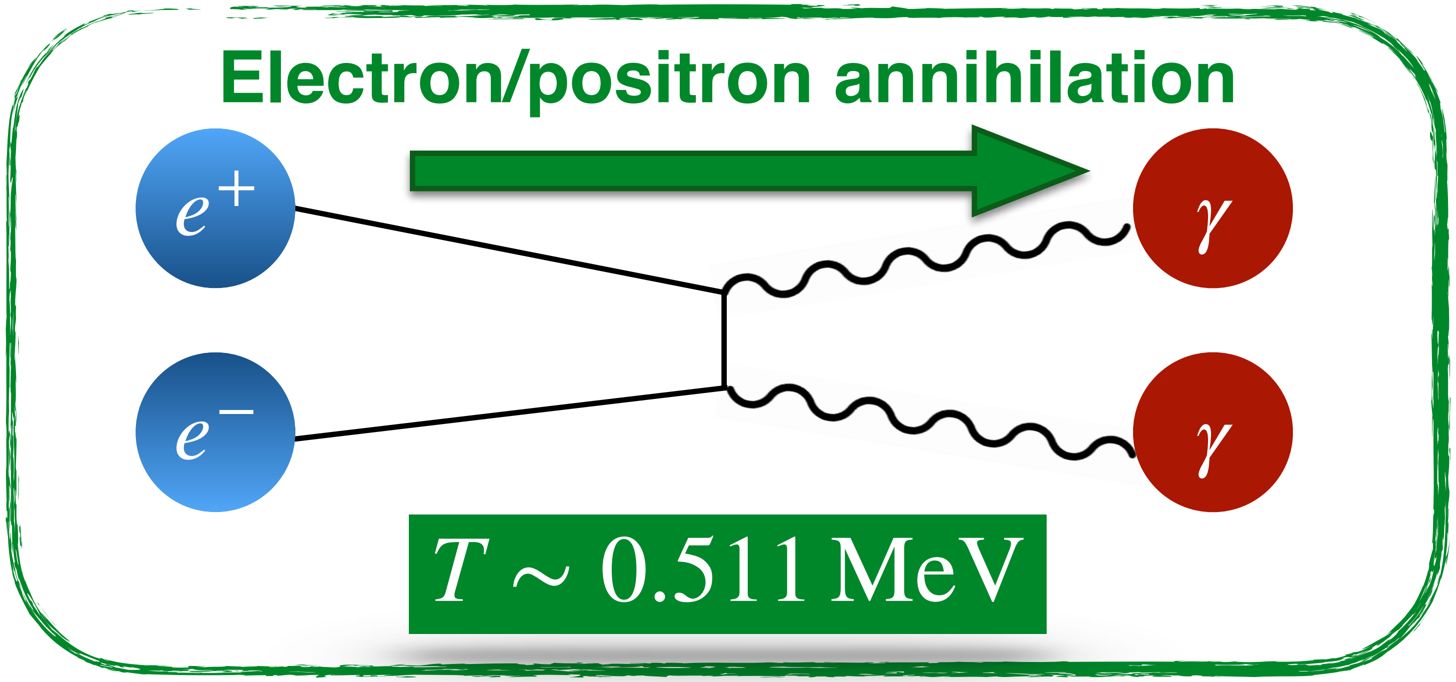


Neutrino decoupling

$T \sim 1\,\mathrm{MeV}$

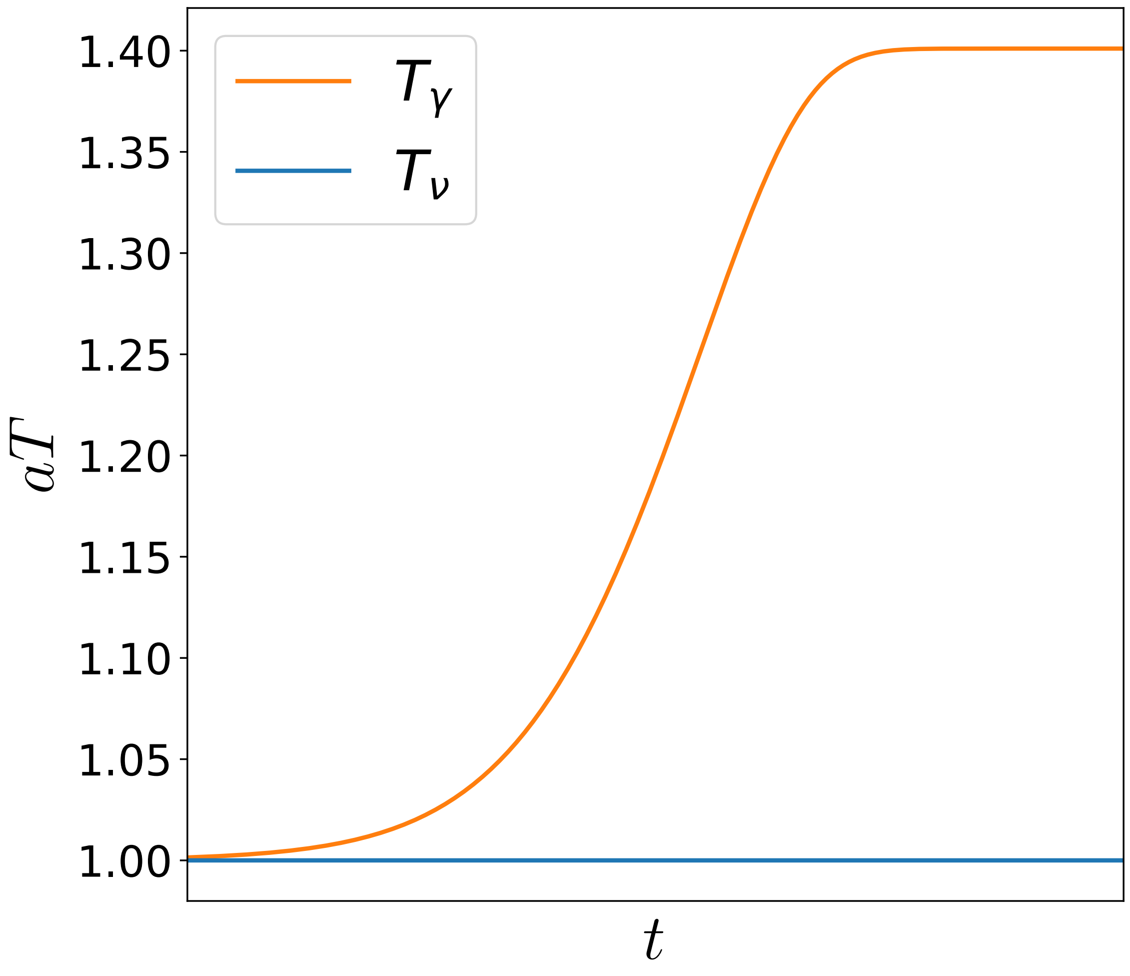
Below this temperature

$$T_{\nu_e} = T_{\nu_\mu} = T_{\nu_\tau} \propto a^{-1}$$



Electron/positron annihilation

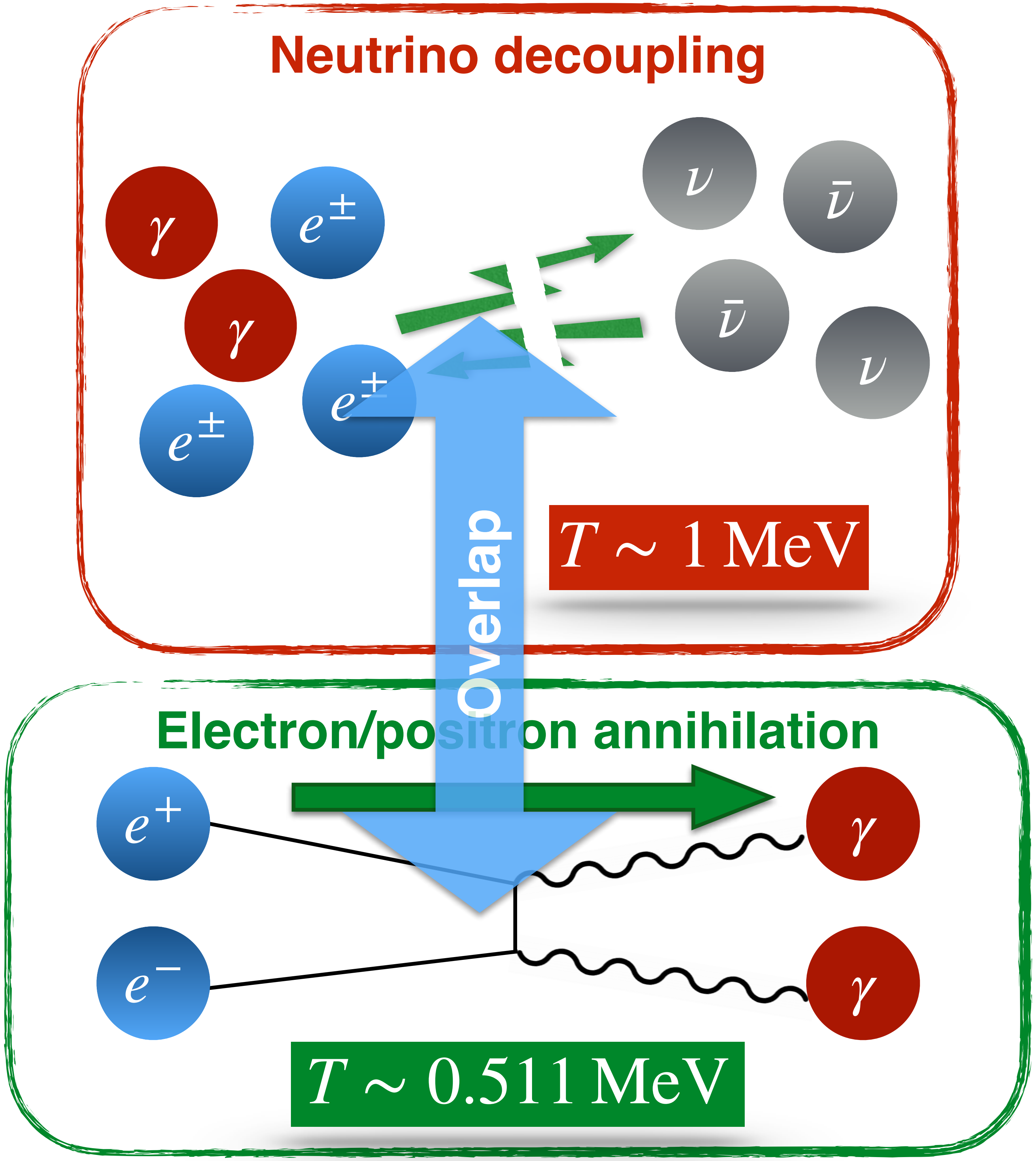
$T \sim 0.511\,\mathrm{MeV}$



Entropy transfer from $e^{\pm}$

T_γ ↘ slower than a^{-1}

Neutrino decoupling



Neutrino decoupling

$T \sim 1\,\mathrm{MeV}$

Overlap

Electron/positron annihilation

$T \sim 0.511\,\mathrm{MeV}$

Below this temperature

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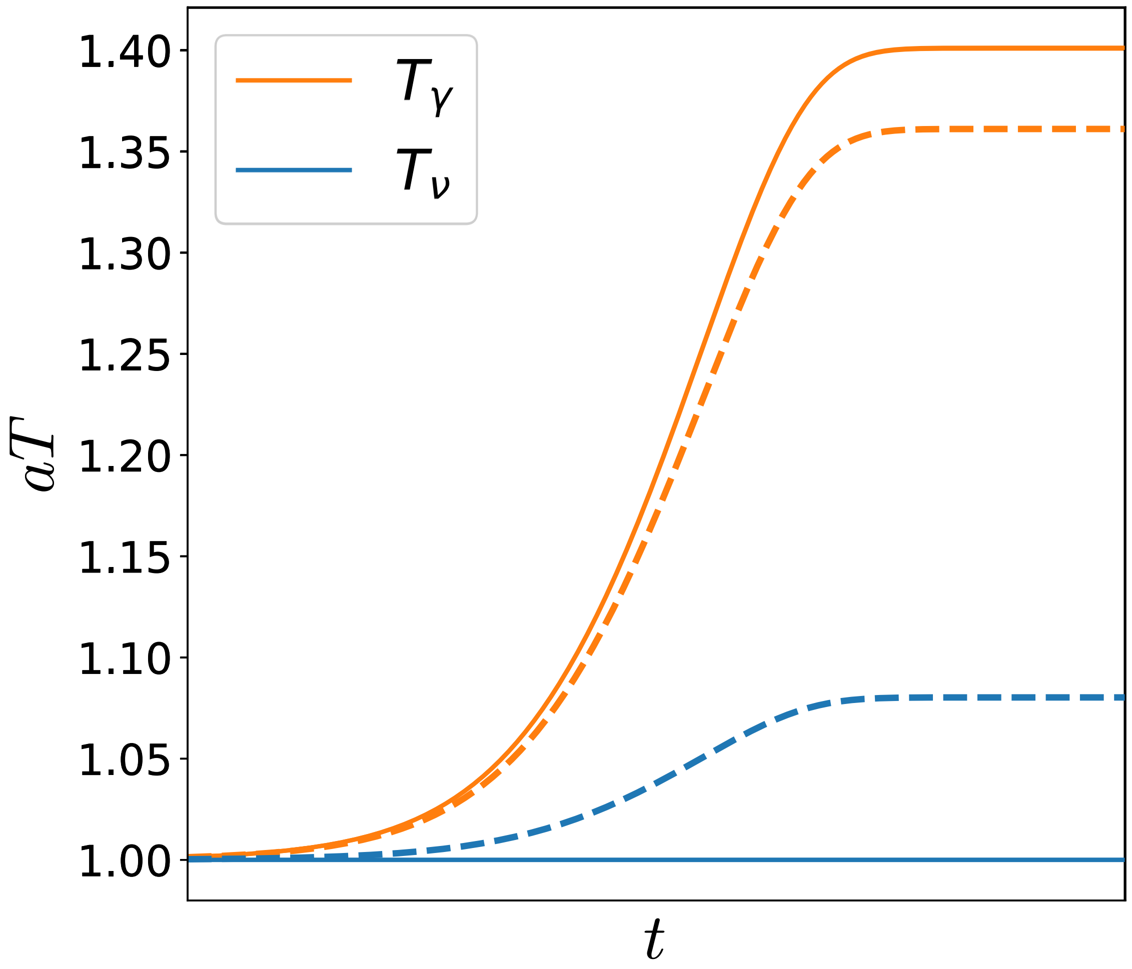
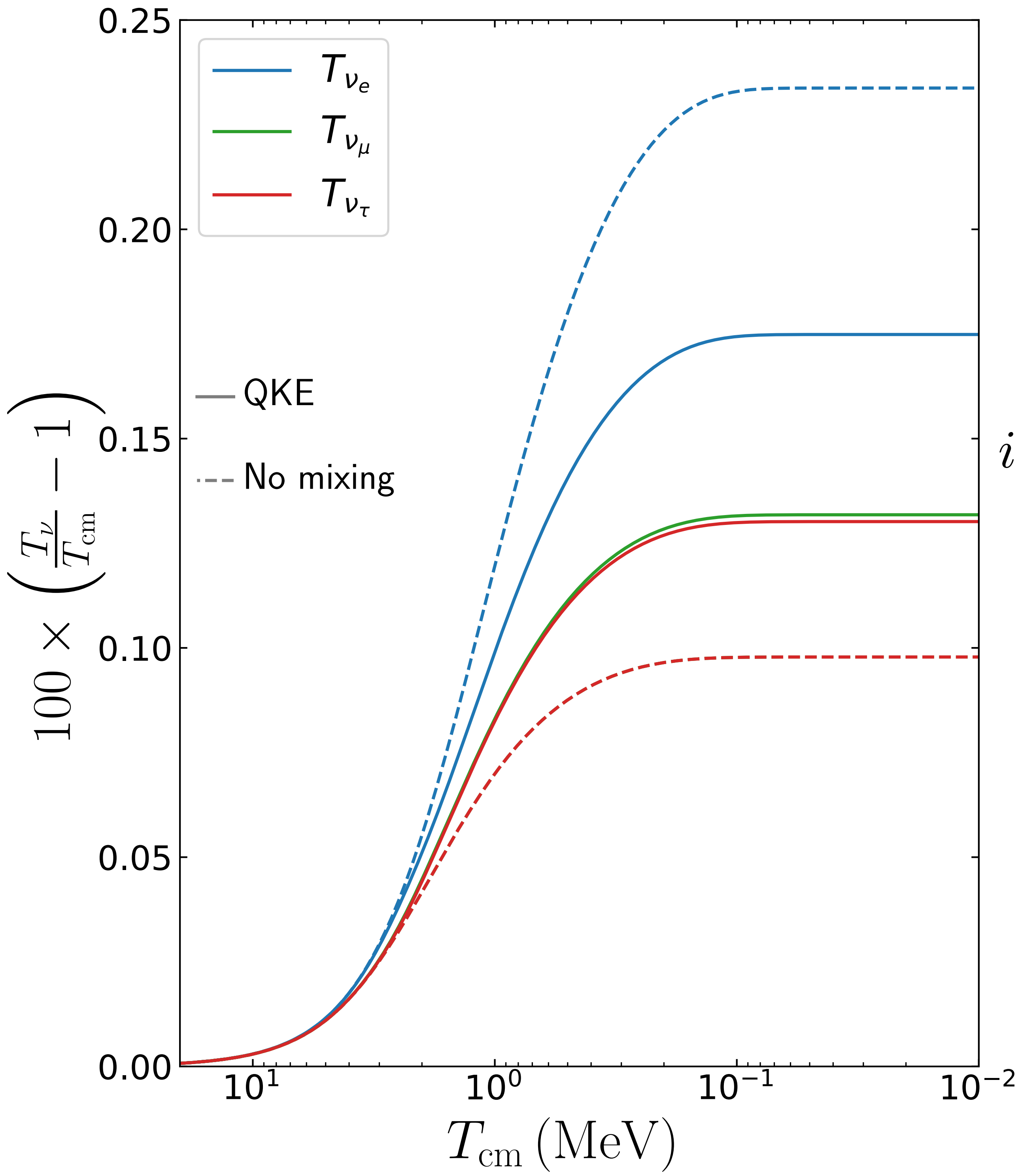


Entropy transfer from $e^{\pm}$

$T_\gamma \searrow$ slower than a^{-1}

Neutrino decoupling with flavor oscillations



Effective temperatures

- Solve the QKE with the hypotheses of homogeneity and isotropy ;
- Species involved: $\nu, \bar{\nu}, e^\pm$

Vacuum

$$i\left[\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}\right]\varrho(p,t) = \left[U\frac{\mathbb{M}^2}{2p}U^\dagger, \varrho\right] - 2\sqrt{2}G_F p\left[\frac{\mathbb{E}_\mathrm{lep} + \mathbb{P}_\mathrm{lep}}{m_W^2}, \varrho\right] + i\,\mathcal{I}[\varrho, \bar{\varrho}]$$

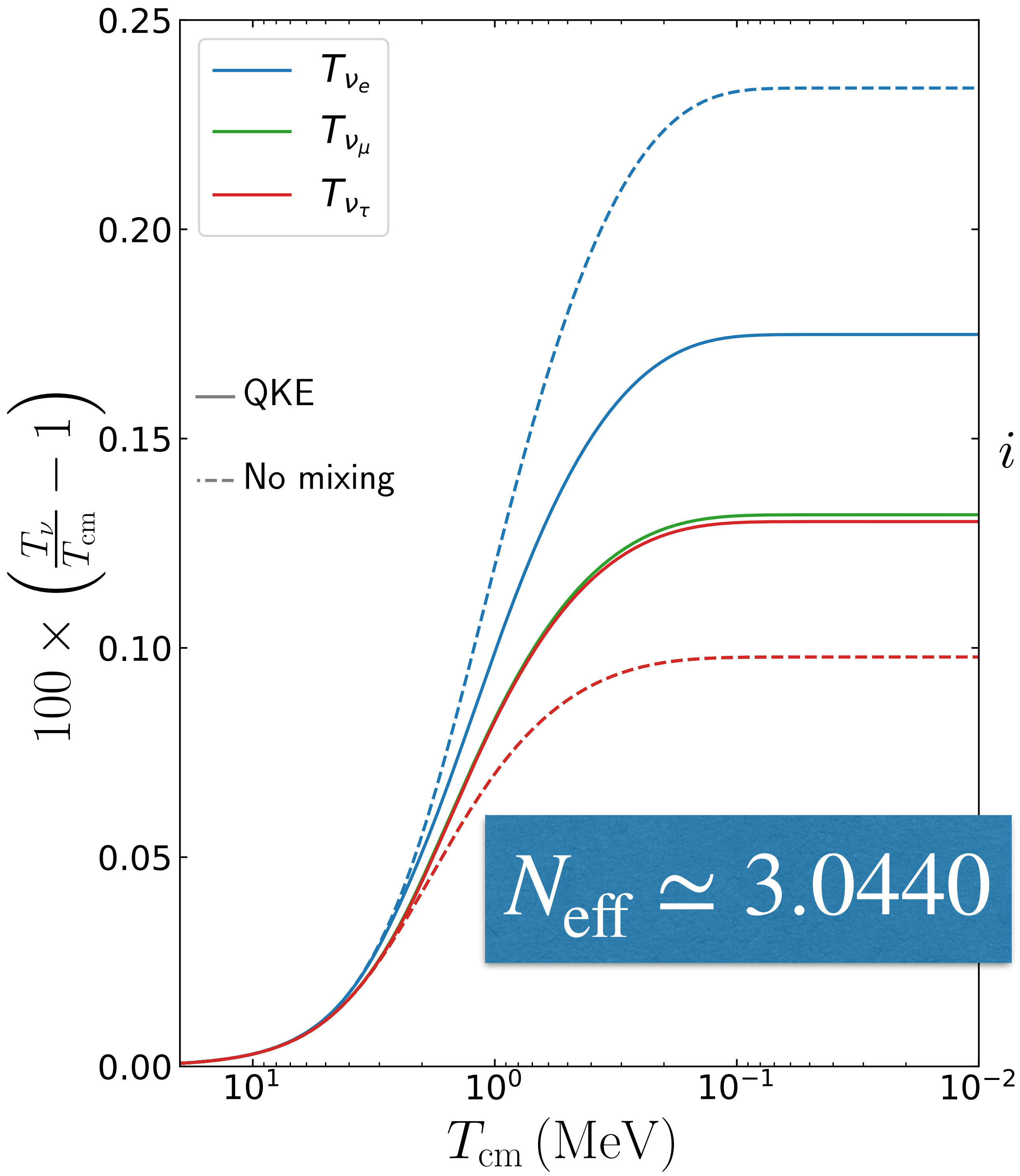
Collisions

Lepton mean-field

- Parameterization:

$$\varrho^\alpha_\alpha(p,t) = \frac{1 + \delta g_{\nu_\alpha}(p,t)}{e^{p/T_{\nu_\alpha}} + 1}$$

Neutrino decoupling with flavor oscillations



$N_{\rm eff} \simeq 3.0440$

Effective temperatures

- Solve the QKE with the hypotheses of homogeneity and isotropy ;
- Species involved: $\nu, \bar{\nu}, e^{\pm}$

$$i\left[\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}\right]\varrho(p,t) = \left[U\frac{\mathbb{M}^2}{2p}U^\dagger, \varrho\right] - 2\sqrt{2}G_F p\left[\frac{\mathbb{E}_{\rm lep} + \mathbb{P}_{\rm lep}}{m_W^2}, \varrho\right] + i\mathcal{I}[\varrho, \bar{\varrho}]$$

Vacuum

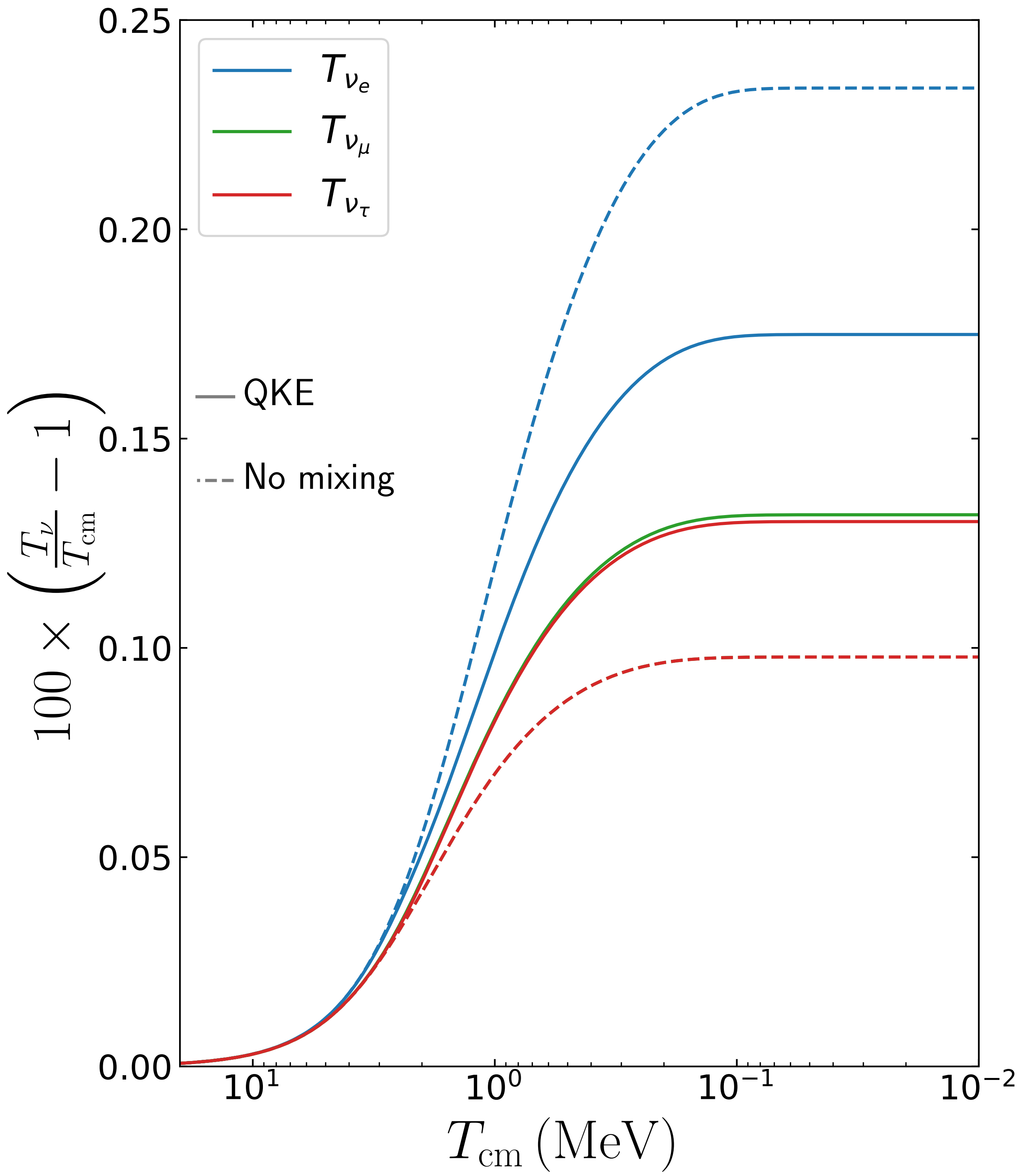
Lepton mean-field

Collisions

- Parameterization:

$$\varrho^\alpha_\alpha(p,t) = \frac{1 + \delta g_{\nu_\alpha}(p,t)}{e^{p/T_{\nu_\alpha}} + 1}$$

Approximation scheme for neutrino oscillations



“No visible oscillations”

$\Longrightarrow$ averaged oscillations?

$\Longrightarrow$ approximate scheme?

Approximation scheme for neutrino oscillations

- Simplified argument: 2-neutrino mixing, no mean-field

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = -i\left[U\frac{\mathbb{M}^2}{2p}U^\dagger, \varrho\right] + \mathcal{I} \quad \Longleftrightarrow \quad \frac{\mathrm{d}\varrho_m}{\mathrm{d}t} = -i\left[\frac{\mathbb{M}^2}{2p}, \varrho_m\right] + U^\dagger \mathcal{I} U$$

$$\varrho_m \equiv U^\dagger \varrho U$$

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$$\varrho_m = \begin{pmatrix} f_1 & a\, e^{i\frac{\Delta m^2}{2p}t} \\ a\, e^{-i\frac{\Delta m^2}{2p}t} & f_2 \end{pmatrix}$$

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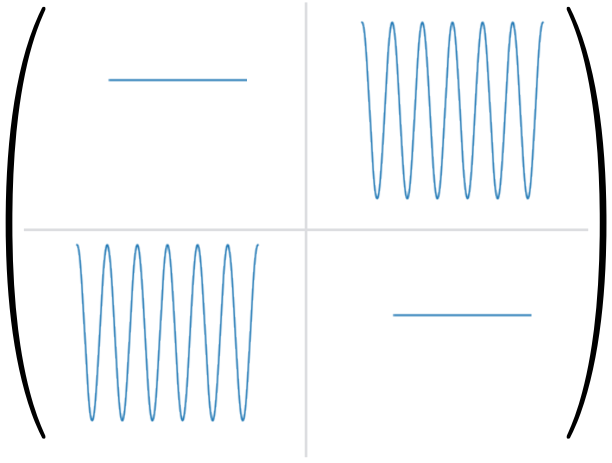
Schematically,

$\varrho_m =$



Localized neutrino injection
$\left(U^\dagger \mathcal{I} U \sim K \times \delta(0)\right)$

Approximation scheme for neutrino oscillations

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$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = -i\Big[U\frac{\mathbb{M}^2}{2p}U^\dagger, \varrho\Big] + \mathcal{I} \quad \Longleftrightarrow \quad \boxed{\frac{\mathrm{d}\varrho_m}{\mathrm{d}t} = -i\Big[\frac{\mathbb{M}^2}{2p}, \varrho_m\Big]} + U^\dagger \mathcal{I} U$$

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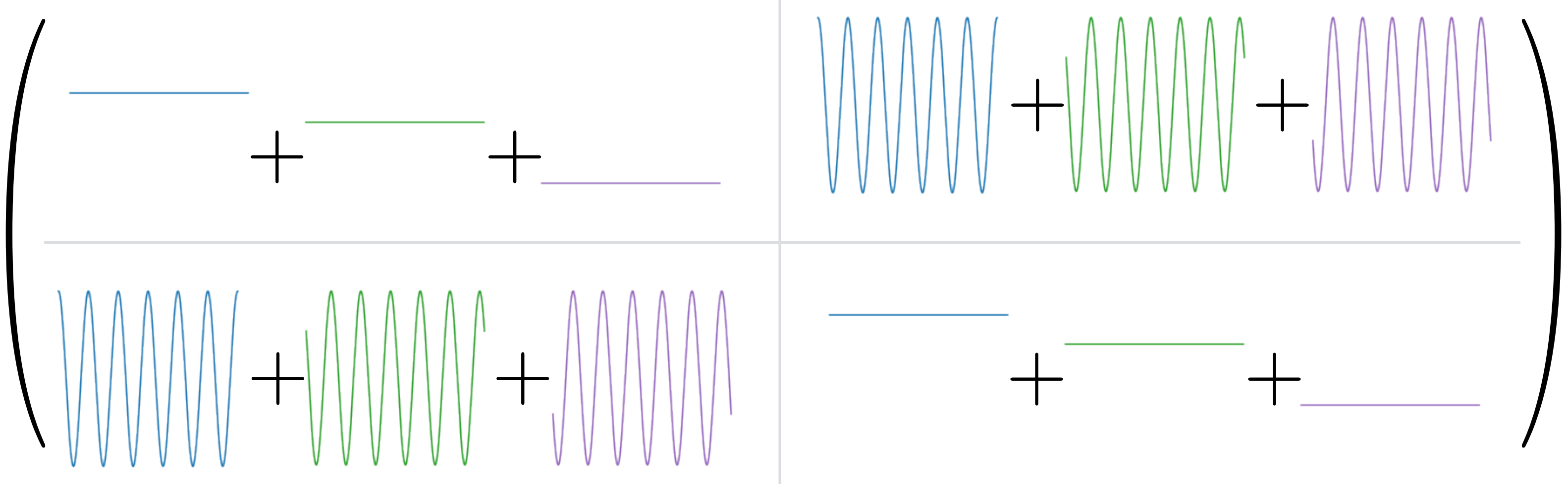

Random neutrino injection

Approximation scheme for neutrino oscillations

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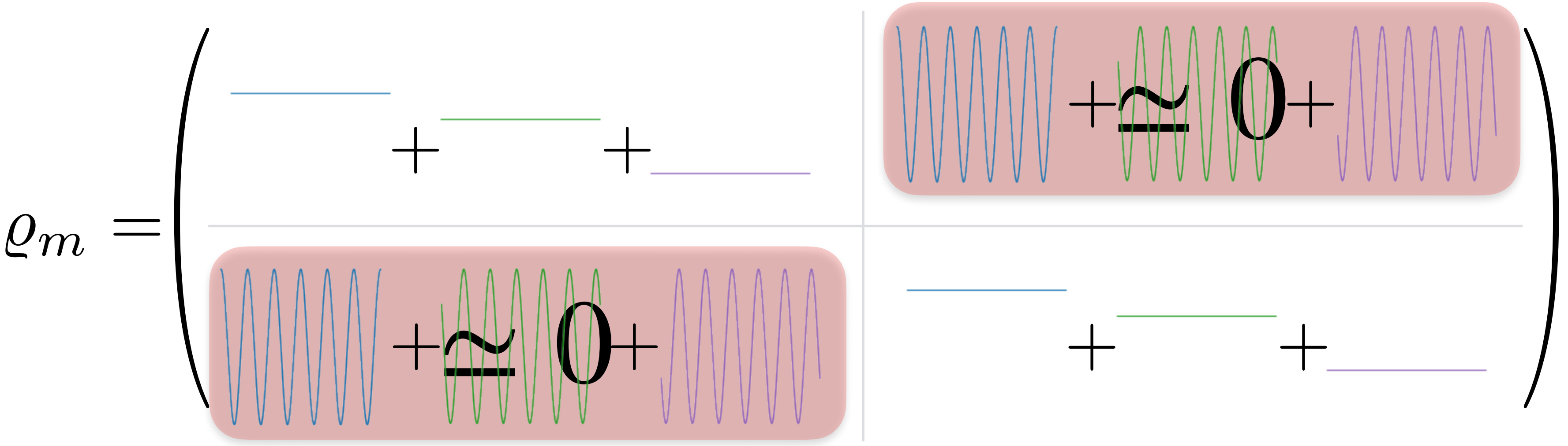

Random neutrino injection

Approximation scheme for neutrino oscillations

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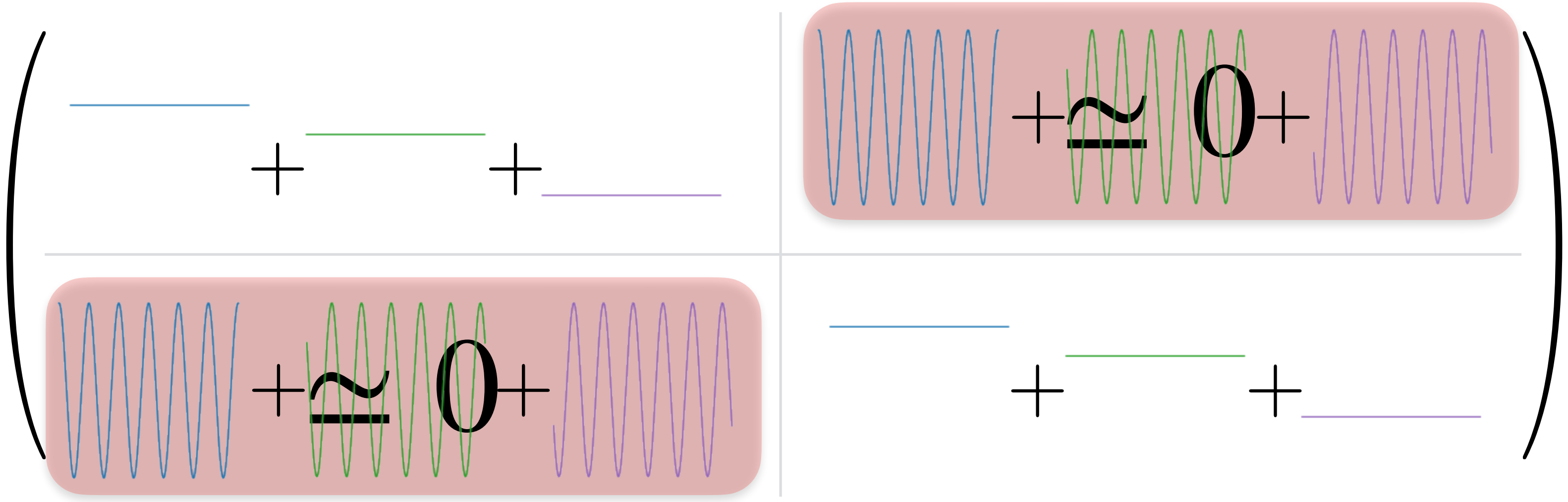
$$\varrho_m \equiv U^\dagger \varrho U$$

Generalization with the full Hamiltonian if it evolves "slowly" (= *adiabatically*).

$\Longrightarrow$ **Adiabatic Transfer of Averaged Oscillations**

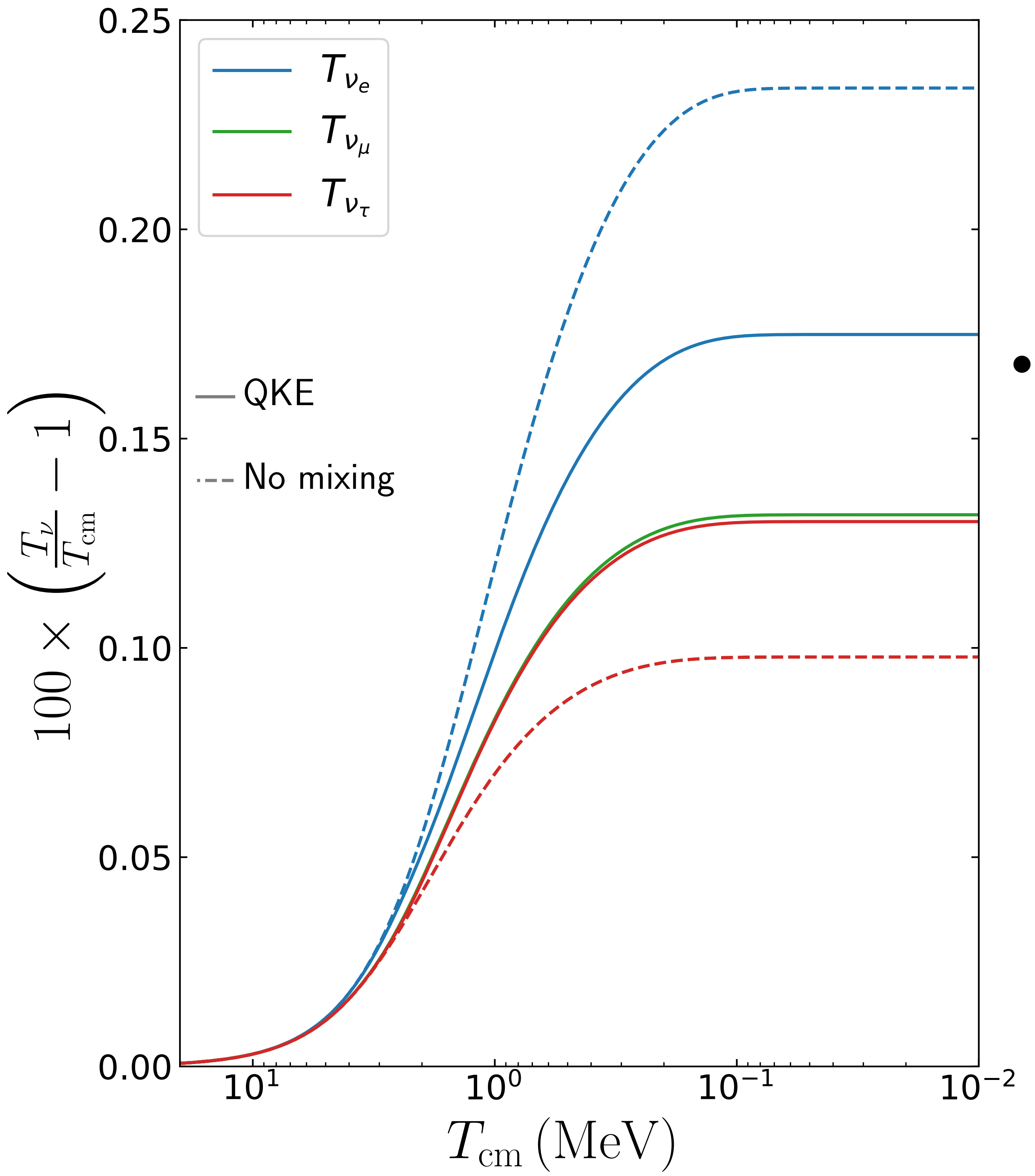
Schematically,

$\varrho_m =$



Random neutrino injection

Neutrino decoupling with flavor oscillations



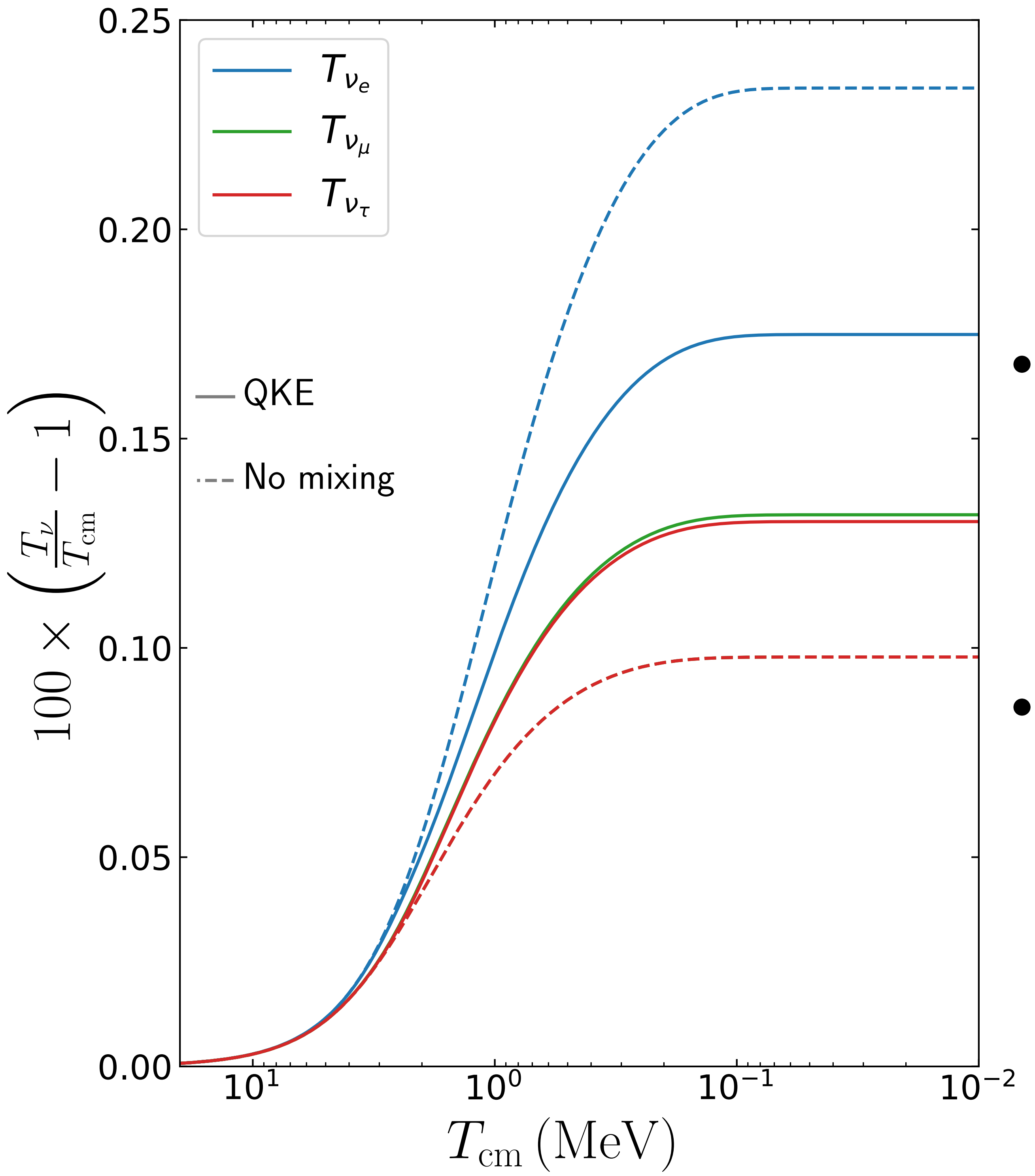
Effective temperatures

Full QKE and ATAO resolutions indistinguishable!
(Difference $\Delta N_{\rm eff} \simeq 10^{-6}$)

- Slight increase of $N_{\rm eff}$ (3.0434 → 3.0440)

  flavor conversion of ν_e $\Longrightarrow$ more phase space for $e^\pm$ annihilations

Neutrino decoupling with flavor oscillations



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- Slight increase of N_{eff} (3.0434 → 3.0440)

  flavor conversion of ν_e $\Longrightarrow$ more phase space for $e^{\pm}$ annihilations

- Deviations from the value $N_{\mathrm{eff}} = 3.0440$:
  - Primordial asymmetries
  - New physics: BSM interactions, sterile neutrinos, …
  - Effect of anisotropies?

Adiabatic Transfer of Averaged Oscillations

$$i\left[\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}\right]\varrho(p,t) = \left[U\frac{\mathbb{M}^2}{2p}U^\dagger, \varrho\right] - 2\sqrt{2}G_F p\left[\frac{\mathbb{E}_{\rm lep}+\mathbb{P}_{\rm lep}}{m_W^2}, \varrho\right] + i\,\mathcal{I}[\varrho,\bar{\varrho}]$$

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→ diagonal in the matter basis,
which evolves ***adiabatically***

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→ diagonal in the matter basis,
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- Non-diagonal components of the density matrix in matter basis are ***averaged out***

$$\varrho_m = \begin{pmatrix} * & \sim & \sim \\ \sim & * & \sim \\ \sim & \sim & * \end{pmatrix} \longrightarrow \tilde{\varrho}_m = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

- Effective “ATAO” equation

$$\boxed{\frac{\partial \tilde{\varrho}_m}{\partial x} = \widetilde{U_m^\dagger \mathcal{K} U_m}}$$

keep only the diagonal

comoving variable $x \propto a$

Quantum Kinetic Equations: collision term

$$i\left[\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}\right]\varrho(p,t) = \left[U\frac{\mathbb{M}^2}{2p}U^\dagger, \varrho\right] - 2\sqrt{2}G_F p\left[\frac{\mathbb{E}_e + \mathbb{P}_e}{m_W^2}, \varrho\right] + i\,\mathcal{C}[\varrho, \bar{\varrho}]$$

Vacuum Mean-field Collisions

$$\boxed{\mathcal{C} = \mathcal{C}^{[\nu e^- \to \nu e^-]} + \mathcal{C}^{[\nu e^+ \to \nu e^+]} + \mathcal{C}^{[\nu\bar{\nu} \to e^- e^+]} + \mathcal{C}^{[\nu\nu]}}$$

$$\begin{aligned}\mathcal{C}^{[\nu e^- \to \nu e^-]} =& \frac{1}{2}\frac{2^5 G_F^2}{2E_1}\int \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3 2E_2}\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3 2E_4} \\ &\times (2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4) \\ &\times \Big[4(p_1\cdot p_2)(p_3\cdot p_4)F_{\mathrm{sc}}^{LL}(\nu^{(1)} + e^{(2)} \to \nu^{(3)} + e^{(4)}) \\ &\quad + 4(p_1\cdot p_4)(p_2\cdot p_3)F_{\mathrm{sc}}^{RR}(\nu^{(1)} + e^{(2)} \to \nu^{(3)} + e^{(4)}) \\ &\quad - 2(p_1\cdot p_3)m_e^2\Big(F_{\mathrm{sc}}^{LR}(\nu^{(1)} + e^{(2)} \to \nu^{(3)} + e^{(4)}) + F_{\mathrm{sc}}^{RL}(\nu^{(1)} + e^{(2)} \to \nu^{(3)} + e^{(4)})\Big)\Big]\end{aligned}$$

Computationally expensive

Quantum Kinetic Equations: collision term

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Computationally expensive

Statistical factor

$$F_{\rm sc}^{AB}(\nu^{(1)}+e^{(2)}\to\nu^{(3)}+e^{(4)})=f_4(1-f_2)\left[G^A\varrho_3G^B(1-\varrho_1)\right]-(1-f_4)f_2\left[G^A(1-\varrho_3)G^B\varrho_1\right]+\text{h.c.}$$

"gain" "loss"

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Computationally expensive

<u>Statistical factor</u>

Pauli-blocking

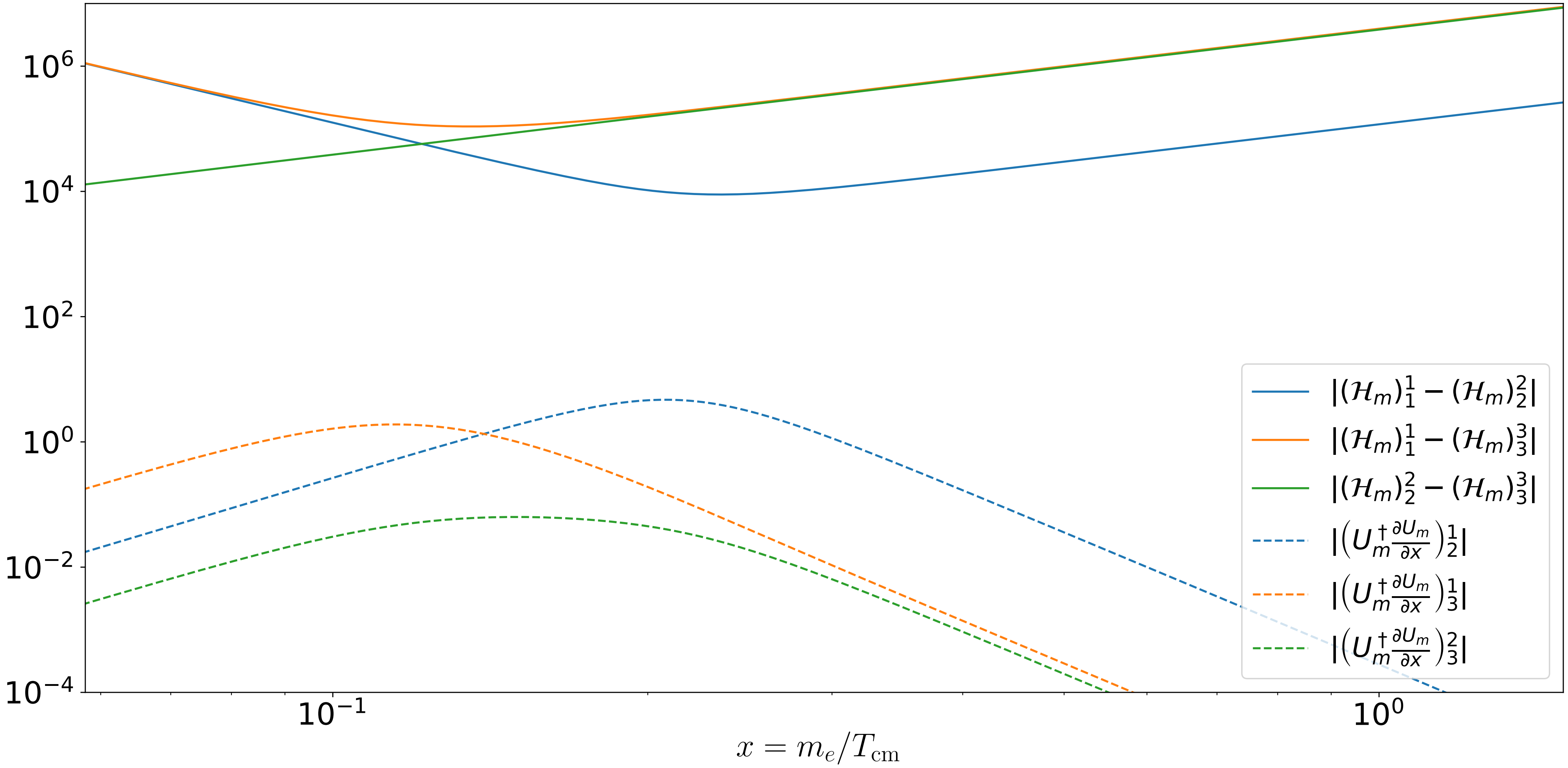
$$F_{\mathrm{sc}}^{AB}(\nu^{(1)}+e^{(2)}\to\nu^{(3)}+e^{(4)}) = f_4(1-f_2)\left[G^A\varrho_3 G^B(1-\varrho_1)\right] - (1-f_4)f_2\left[G^A(1-\varrho_3)G^B\varrho_1\right] + \mathrm{h.c.}$$

"gain" "loss"

Checking the adiabatic approximation

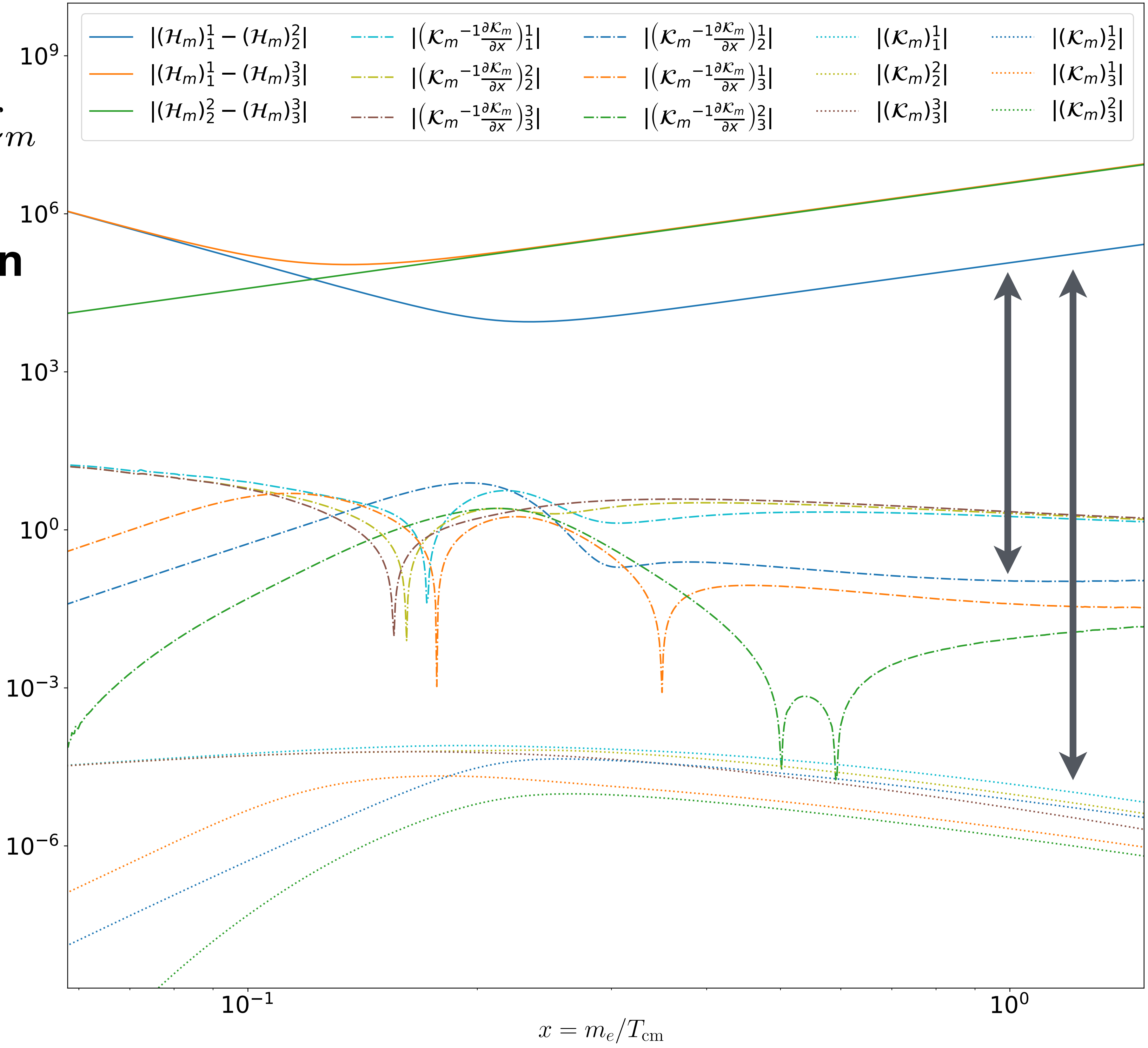
$$\frac{\partial \varrho_m}{\partial x} = -i[\mathcal{H}_m, \varrho_m] - [U_m^\dagger \frac{\partial U_m}{\partial x}, \varrho_m] + \mathcal{K}_m$$

Effective oscillation frequencies



Checking that oscillations are averaged

$$\frac{\partial \varrho_m}{\partial x} = -i[\mathcal{H}_m, \varrho_m] + \mathcal{K}_m$$



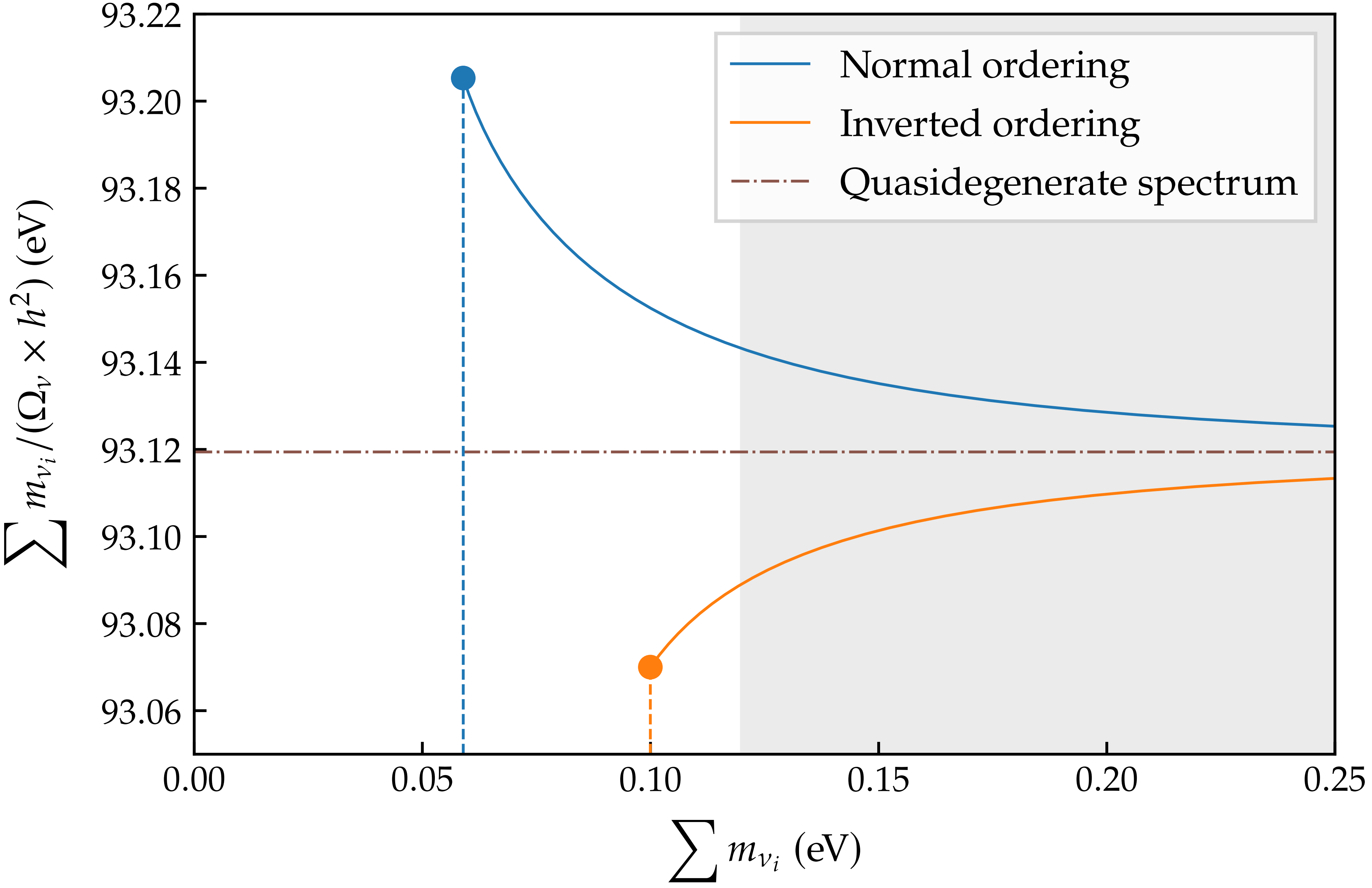
Effective oscillation frequencies

Relative variation of collision term

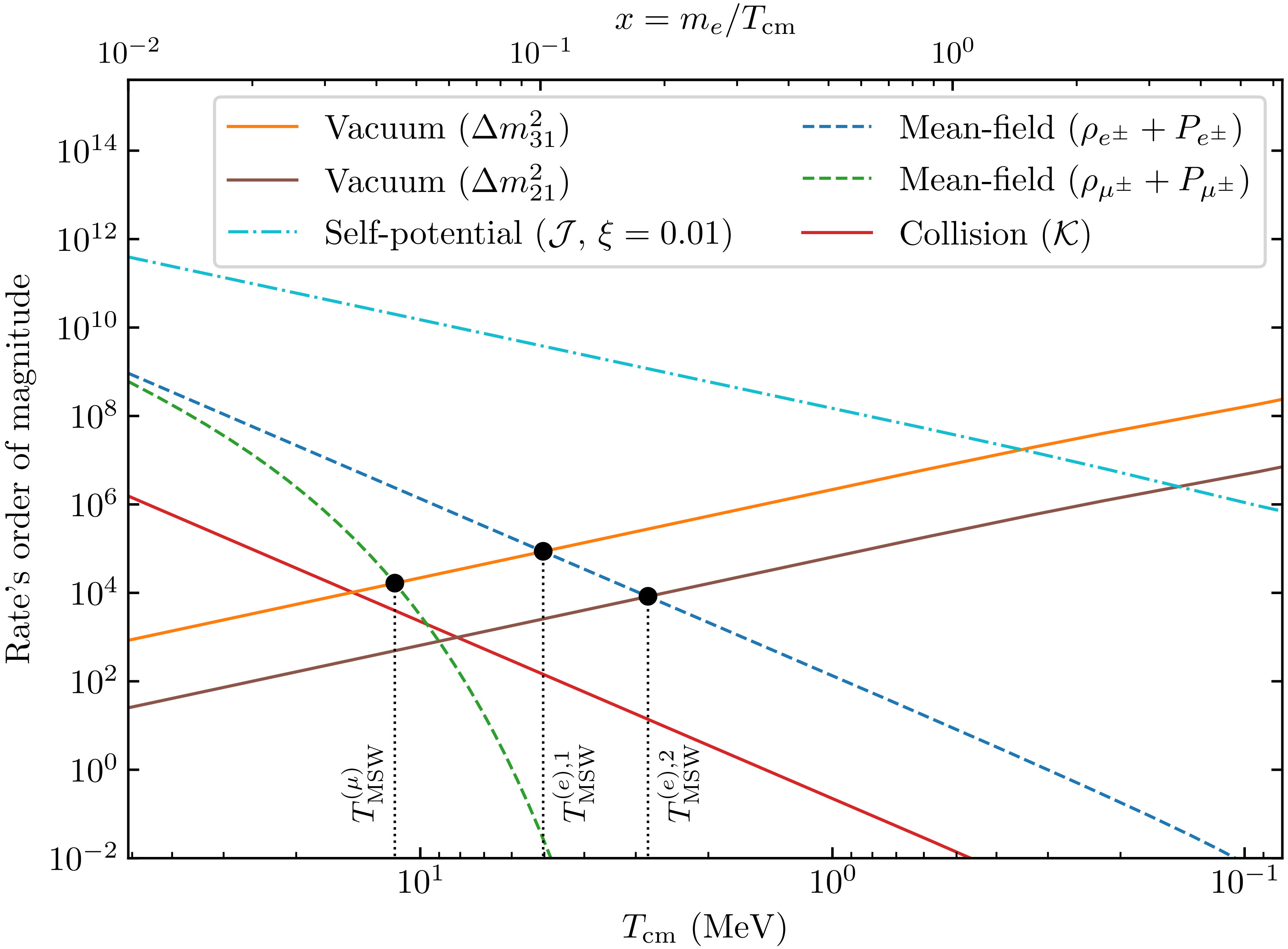
Collision rate

Using the results from neutrino decoupling

- Neutrino energy density today $$\Omega_\nu \equiv \frac{\rho_\nu + \rho_{\bar{\nu}}}{\rho_\text{crit}} = \frac{2\sum m_{\nu_i} n_{\nu_i}}{\rho_\text{crit}}$$

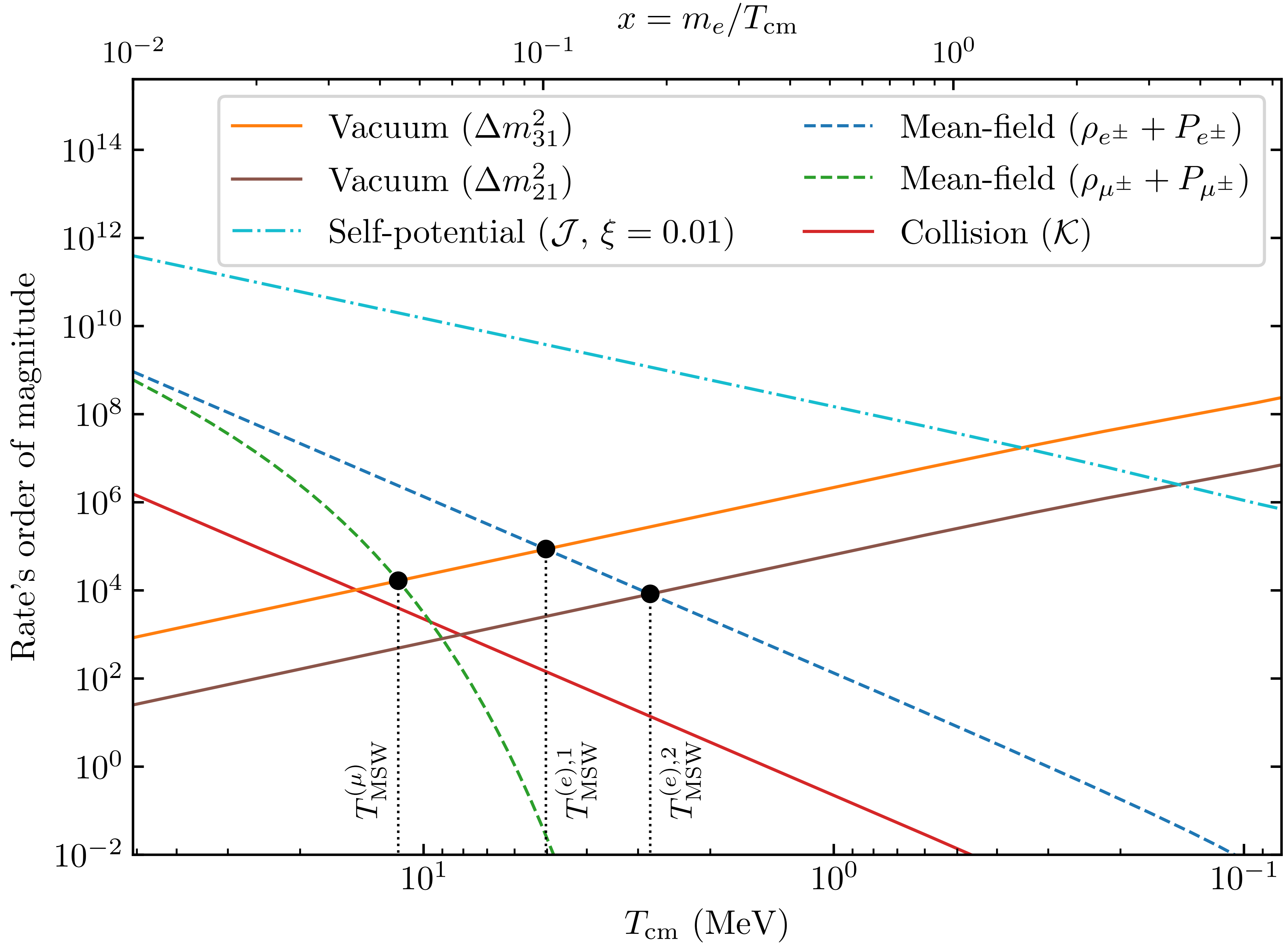


Comparison of Hamiltonian contributions



$$\mathcal{H}_0 \propto x^2 \times U \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta m^2_{21} & 0 \\ 0 & 0 & \Delta m^2_{31} \end{pmatrix} U^\dagger \qquad \mathcal{H}_{\rm lep} \propto x^{-4} \times \begin{pmatrix} \rho_{e^\pm} + P_{e^\pm} & 0 & 0 \\ 0 & \rho_{\mu^\pm} + P_{\mu^\pm} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Comparison of Hamiltonian contributions



MSW transition when $\mathcal{H}_{\rm lep}$ domination becomes $\mathcal{H}_0$ domination

“Moment” Quantum Kinetic Equations

- Angular moments of the density matrix:

$$\begin{bmatrix} N \\ F^i \\ P^{ij} \end{bmatrix} = p^2 \int \mathrm{d}\Omega \begin{bmatrix} 1 \\ p^i/p \\ p^i p^j/p^2 \end{bmatrix} \varrho(t, \vec{x}, \vec{p})$$

- Focus on “fast-flavor instabilities”, governed by the Hamiltonian:

$$\mathcal{H}_{\mathrm{self}} = \frac{\sqrt{2}G_F}{(2\pi)^3} \int \mathrm{d}^3\vec{q}\,(1-\cos\theta)\left[\varrho(t,\vec{x},\vec{q}) - \bar{\varrho}(t,\vec{x},\vec{q})\right]$$

- QKEs for moments:

$$i\left(\frac{\partial N}{\partial t} + \frac{\partial F^j}{\partial x^j}\right) = \sqrt{2}G_F\left[N-\overline{N}, N\right] - \sqrt{2}G_F\left[(F-\overline{F})_j, F^j\right]$$

$$i\left(\frac{\partial F^i}{\partial t} + \frac{\partial P^{ij}}{\partial x^j}\right) = \sqrt{2}G_F\left[N-\overline{N}, F^i\right] - \sqrt{2}G_F\left[(F-\overline{F})_j, P^{ij}\right]$$

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- QKEs for moments:

Closure $P^{ij}_{\alpha\beta}\left(N_{\alpha\beta}, F^k_{\alpha\beta}\right)$

$$i\left(\frac{\partial N}{\partial t} + \frac{\partial F^j}{\partial x^j}\right) = \sqrt{2}G_F\left[N-\overline{N}, N\right] - \sqrt{2}G_F\left[(F-\overline{F})_j, F^j\right]$$

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